Complex Analysis

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Lecture – 4

Topology on the Complex Plane(Continued)

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Compactness

Let (X, d) be a metric space. Let $\mathscr{U}$ be a collection of open sets in X. We say that $\mathscr{U}$ is an open cover of a subset $K \subset X$ if
$$K \subset \bigcup \{U : U \in \mathscr{U}\}.$$

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We say that a subset $K \subset X$ is compact if for every open cover $\mathscr{U}$ of K, $\exists$ $U_1, U_2, \ldots, U_n \in \mathscr{U}$ s.t.
$$K \subset \bigcup_{k=1}^{n} U_k.$$

We say that a subset $K \subseteq X$ is **compact** if for every open cover $\mathscr{U}$ of K, there exists finitely many elements $U_1, U_2, \ldots U_n \in \mathscr{U}$ such that $K \subseteq \bigcup_{k=1}^{n} U_k$.

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Example: * A finite subset of a metric space is compact

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Example:

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- $\varnothing$ is compact.
- Let $\{x_n\}$ be a sequence of points converging to x_0. Then $A = \{x_n : n \in \mathbb{N} \cup \{0\}\}$ is compact in X.

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Then $A = \{x_n : n \in \mathbb{N} \cup \{0\}\}$ is compact in X.

Proposition: In a metric space, a compact set is closed.

Proof: Let $K \subseteq X$ be compact & $x_0 \in X \setminus K$.

Define $B_n := \{x \in X : d(x, x_0) \leq 1/n\}$

PROPOSITION 1. *In a metric space, a compact set is closed.*

PROOF. Let $K \subseteq X$ be compact and $x_0 \in X \setminus K$. Define $B_n := \{x \in X : d(x, x_0) \leq \frac{1}{n}\}$. Let $\mathscr{U} = \{U_n\}_{n \in \mathbb{N}}$, where $U_n = X \setminus B_n$.

Then $X \setminus (\bigcup_{n=1}^{\infty} U_n) = \bigcap_{n=1}^{\infty} (X \setminus U_n) = \bigcap_{n=1}^{\infty} B_n = \{x_0\}$. Since we choose x_0 not in $K \implies K \subseteq \bigcup_{n=1}^{\infty} U_n$. Hence $\mathscr{U}$ is an open cover of K. Since K is compact, there exists a finite subcover of $\mathscr{U}$. i.e, $\exists n_0 \in \mathbb{N}$ such that $K \subseteq U_1 \cup U_2 \cup \cdots \cup U_{n_0}$.

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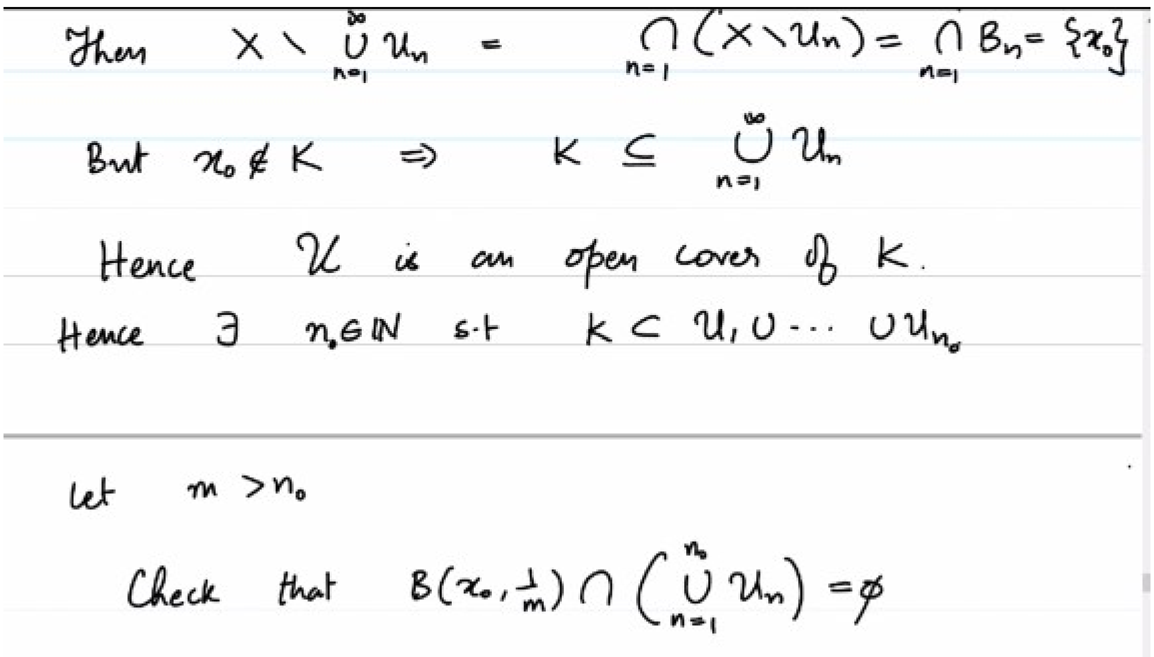

Let $m > n_0$, now reader can verify that $B(x_0, \frac{1}{m}) \cap (\bigcup_{n=1}^{n_0} U_n) = \varnothing \implies B(x_0, \frac{1}{m}) \subset X \setminus K$. Hence $X \setminus K$ is open $\implies K$ is closed.

$\square$

EXERCISE 2. A closed subset of a compact set is closed.

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Let X be a metric space. We say that X is limit point compact if every infinite subset of X has a limit point.

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PROPOSITION 3. *Let X be a compact metric space, then X is limit point compact.*

PROOF. Let A be an infinite subset of X. Suppose A does not have a limit point, then A is closed in $X \implies X \setminus A$ is open.

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Suppose A does not have a limit point, then A is closed in X. $\Rightarrow$ $X \setminus A$ is open.

Given $x \in A$, $\exists$ a nghd U_x of A s.t.
$U_x \cap A = \{x\}$.
Define $\mathcal{U} := \{U_x : x \in A\} \cup \{X \setminus A\}$.

Given $x \in A$, then x is not a limit point of A. Hence $\exists$ a neighborhood U_x of x such that $A \cap U_x = \{x\}$. Define $\mathscr{U} := \{U_x : x \in A\} \bigcup \{X \setminus A\}$. Then $\mathscr{U}$ is an open cover of X. Since X is compact $\exists\, x_1, x_2, \ldots x_n \in A$ such that $X = U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n} \cup (X \setminus A)$.

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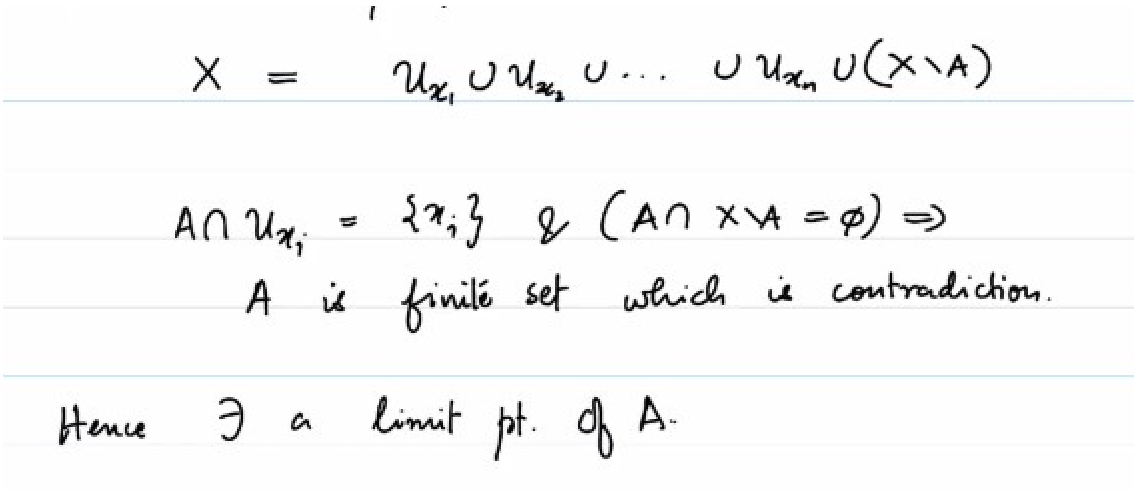

But $A \cap U_{x_i} = \{x_i\}$ and $A \cap (X \setminus A) = \varnothing \implies A$ is finite which is a contradiction. Hence $\exists$ a limit point of A. $\square$

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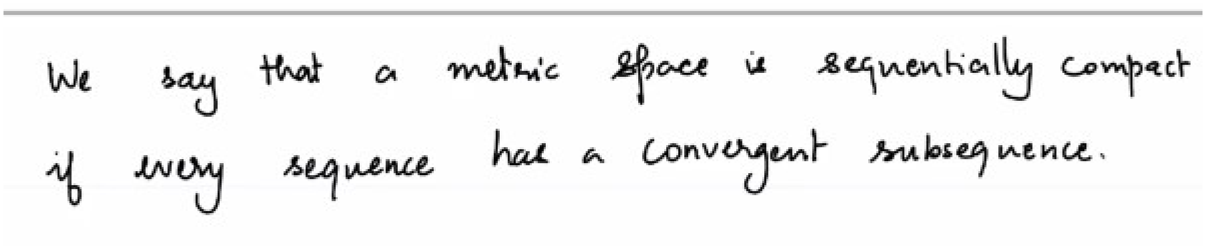

We say that metric space is **sequentially compact** if every sequence in the metric space has that convergent subsequence.

PROPOSITION 4. *Let X be a metric space. If X is limit point compact, then X is sequentially compact.*

PROOF. Let $\{x_n\}$ be a sequence in X and $A = \{x_n : n \in \mathbb{N}\}$. Then A can either be finite or infinite.

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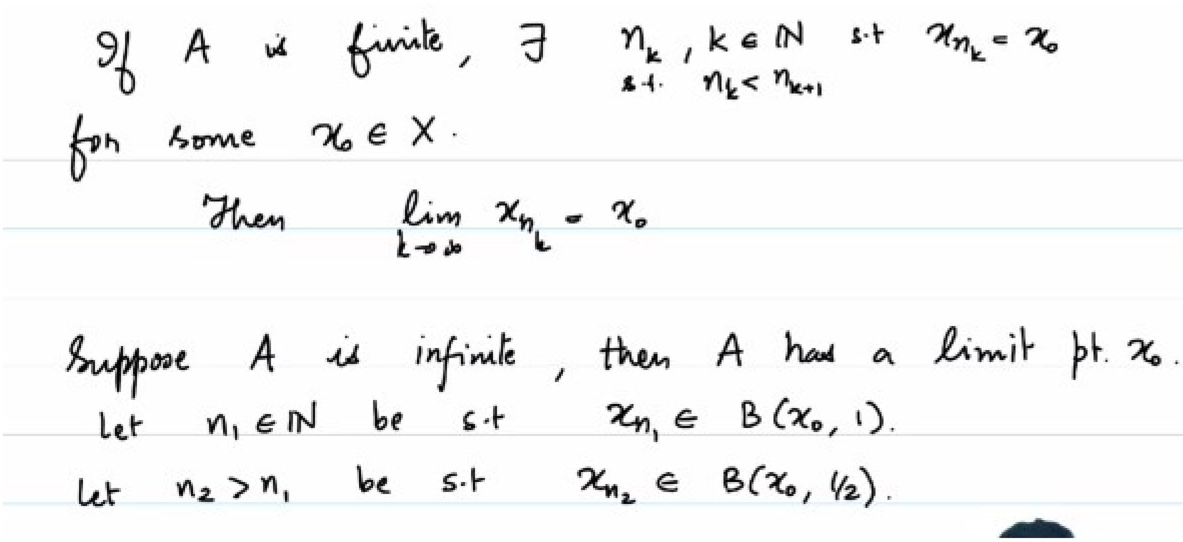

If A is finite, $\exists n_k$ such that $n_k < n_{k+1}, k \in \mathbb{N}$ and $x_{n_k} = x_0$ for some $x_0 \in X$ which repeats infinitely many times in the sequence. Then $\lim_{k \to \infty} x_{n_k} = x_0$ Hence $\{x_n\}$ has a convergent subsequence. If A is infinite, then A has a limit point x_0, since X is limit point compact.

Claim: There exists a subsequence of $\{x_n\}$ converging to x_0.

Let $n_1 \in \mathbb{N}$ be such that $x_{n_1} \in B(x_0, 1)$. Let $n_2 > n_1$ be such that $x_{n_2} \in B(x_0, \frac{1}{2})$.

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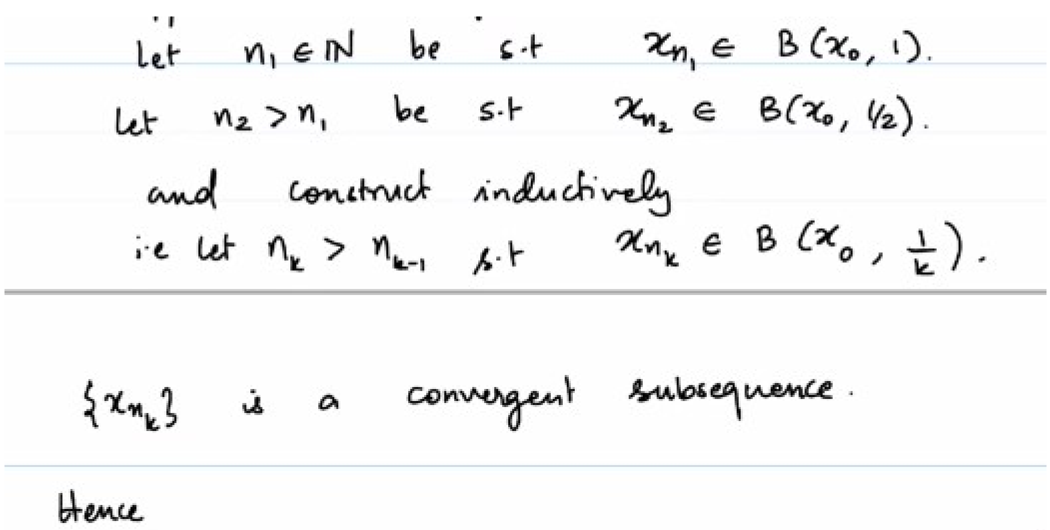

Reader should verify the existence of such an n_2.(*Hint: If there does not exists such an n_2 such that $n_2 > n_1$ and $x_{n_2} \in B(x_0, \frac{1}{2})$. Then $d(x_k, x_0) > \frac{1}{2}$ $\forall k > n_1$. Let $\delta = min\{d(x_0, x_1), d(x_0, x_2), \ldots d(x_0, x_{n_1}), \frac{1}{2}\}$. Then consider $B(x_0, \delta)$.*)

Construct inductively x_{n_k} such that $n_k > n_{k-1}$ and $x_{n_k} \in B(x_0, \frac{1}{k})$. Then $\{x_{n_k}\}$ converges to x_0. Hence $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$. Hence X is sequentially compact.

$\square$

Now we have proved that if a metric space is compact then it is limit point compact. If a metric space is limit point compact then it is sequentially compact. We can complete 'the cycle' by proving that any sequentially compact metric space is a compact metric space. For that we need to use a theorem called Lebesgue number lemma.

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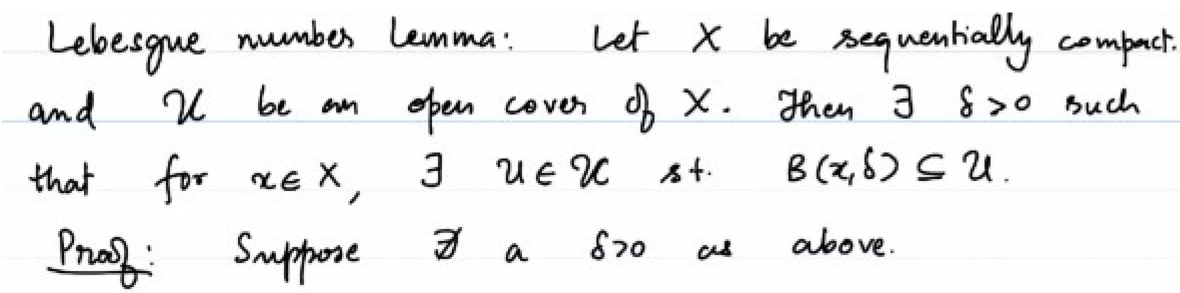

LEMMA 5 (**Lebesgue number lemma**). *Let X be sequentially compact and $\mathcal{U}$ be an open cover of X. Then $\exists \delta > 0$ such that for $x \in X$, $\exists U \in \mathcal{U}$ such that $B(x, \delta) \subset U$.*

PROOF. Suppose there does not exists such a $\delta > 0$. Then for every $n \in \mathbb{N}$, $\exists x_n \in X$ such that $B(x_n, \frac{1}{n})$ is not contained in any element of $\mathcal{U}$. Since X is sequentially compact, $\exists$ a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \longrightarrow x_0, x_0 \in X$. $\mathcal{U}$ is an open cover of $X, x_0 \in X \implies \exists \epsilon > 0$ such that $B(x_o, \epsilon) \subset U$ for some $U \in \mathcal{U}$. Let n_k be large enough such that $\frac{1}{n_k} < \frac{\epsilon}{2}$ and $d(x_{n_k}, x_o) < \frac{\epsilon}{2}$.

Claim: $B(x_{n_k}, \frac{1}{n_k}) \subset U$.

Let $y \in B(x_{n_k}, \frac{1}{n_k})$, then $d(x_0, y) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y) < \epsilon \implies y \in B(x_0, \epsilon) \subset U \implies B(x_{n_k}, \frac{1}{n_k}) \subset U$.

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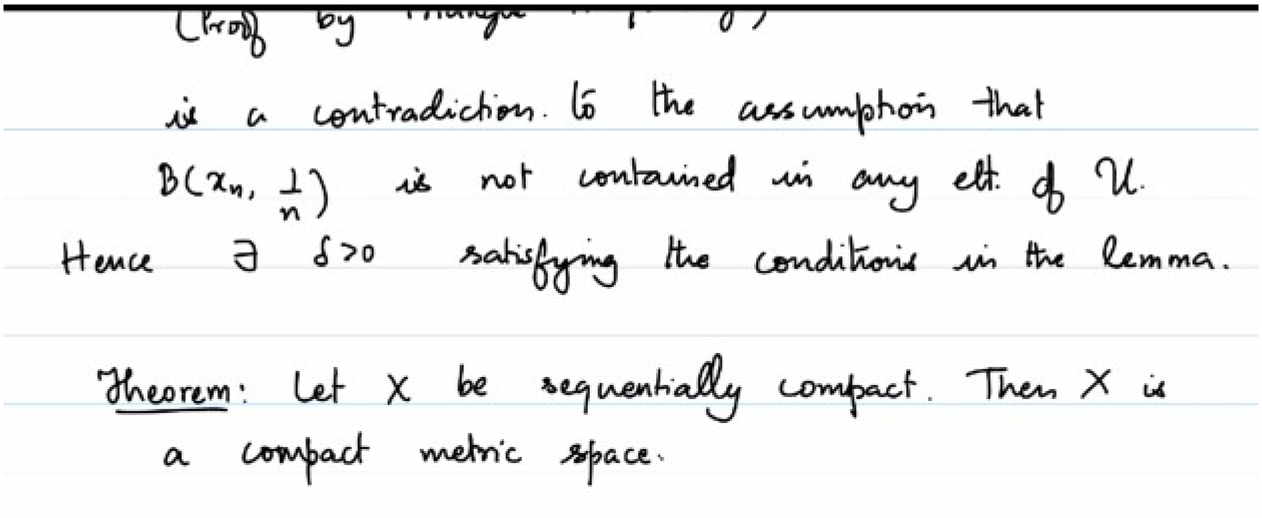

This is a contradiction to the assumption that $B(x_n, \frac{1}{n})$ is not contained in any element of $\mathscr{U}$. Hence $\exists \delta > 0$ satisfying the condition in the lemma. □

THEOREM 6. *Let X be sequentially compact. Then X is compact metric space.*

PROOF. **Claim:** Given $\epsilon > 0, \exists$ finitely many points $x_1, x_2, \ldots x_n$ such that $X = B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$.

Suppose $\epsilon > 0$ is such that there does not exists finitely many points $x_1, x_2, \ldots x_n$ such that $X = B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$. Pick $x_1 \in X$. Inductively $x_{n+1} \notin B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$. Consider the sequence $\{x_n\}$. But X is sequentially compact, hence $\exists$ a converging subsequence of $\{x_n\}$. But from our assumption $d(x_j, x_i) > \epsilon$ and hence no subsequence will be Cauchy, which is a contradiction.

Hence $\exists$ finitely many points $x_1, x_2, \ldots x_n$ such that $X = B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$ (This particular notion sometimes also called as total boundedness).

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$X = B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$.

Let $\mathcal{U}$ be an open cover of X. By lebesgue number lemma, $\exists$ $\delta > 0$ s.t. given x, $B(x, \delta) \subset U_x$ for some $U_x \in \mathcal{U}$.

By above claim, $\exists$ finitely many pts. s.t

$B(x_1, \delta) \cup \cdots \cup B(x_n, \delta) = X$.

Let $\mathscr{U}$ be an open cover of X. Then by using Lebesgue number lemma, $\exists \delta > 0$ such that given $x \in X, B(x, \delta) \subset U_x$ for some $U_x \in \mathscr{U}$. By the above claim $\exists$ finitely many points such that $X = B(x_1, \delta) \cup B(x_2, \delta) \cup \cdots \cup B(x_n, \delta) \implies X \subset U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$. Hence X is compact. $\square$

With this, we complete our circle of ideas. From now on, because our complex plane is metric space with respect to the metric we have define, these notions coincide in our complex plane and we will freely be using these notions interchangeable.