Complex Analysis

Prof. Pranav Haridas

Kerala School of Mathematics

Lecture No – 38

Residue Theorem

Let us explore the behavior of holomorphic functions around its isolated singularities a bit further. Here we will state and prove the residue theorem. Residue theorem broadly answers the following question: suppose you have a function f which is holomorphic on an open set $\Omega \setminus S$, where S is a discrete set of singularities of f, then what can we say about $\int_\gamma f(z)dz$, where γ is some closed curve in $\Omega \setminus S$.

Suppose S was empty and γ was null homotopic, then this is just the Cauchy's theorem. The Cauchy's theorem tells us that this integral is equal to 0. So in some sense, the residual theorem can be thought of as a generalization of the Cauchy's theorem. However the presence of these singularities makes the answer to this question a bit more complicated.

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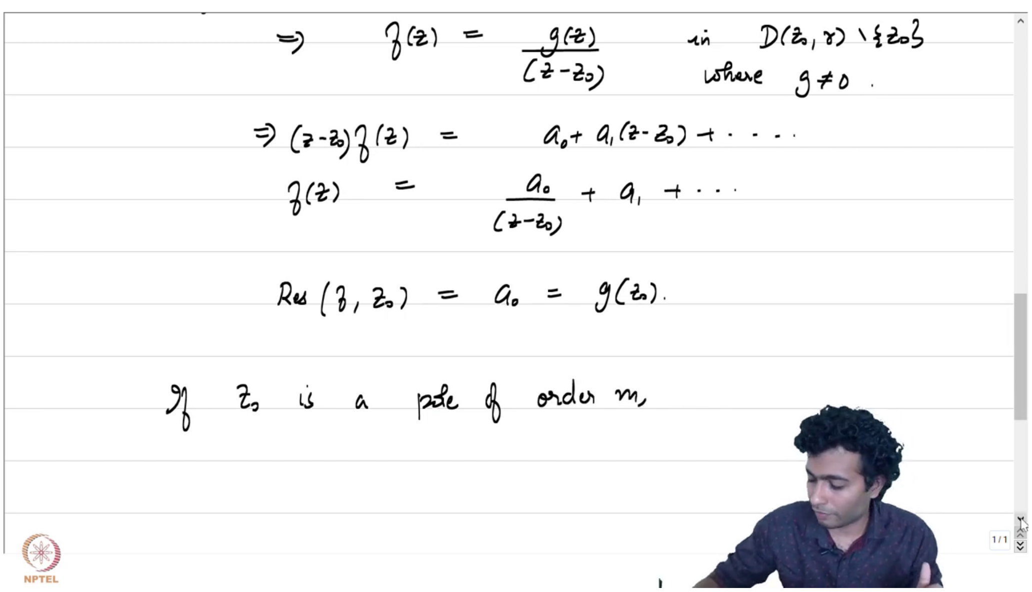

DEFINITION 1 (Residue of a function at a point z_0). Let $f : \Omega \setminus S \longrightarrow \mathbb{C}$ be a holomorphic function where Ω is an open set and S is a discrete subset of Ω. For $z_0 \in S$, let $r > 0$ be such that $\overline{D(z_0, r)} \subseteq \Omega$ and $D(z_0, r) \cap S = \{z_0\}$. Then in $D(z0, r) \setminus \{z_0\}$, consider the Laurent series expansion of f given by,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

We define the residue of f at z_0 to be

$$\text{Res}\left(f, z_0\right) := a_{-1}.$$

If z_0 is a removable singularity, then $a_{-n} = 0$ for every $n \in \mathbb{N}$. Hence $\text{Res}\left(f, z_0\right) = 0$.

If z_0 is a pole of f of order 1, then

$$f(z) = \frac{g(z)}{(z - z_0)} \qquad \text{in } D(z_0, r) \setminus \{z_0\}$$

where $g \neq 0$. Thus, we have,

$$(z - z_0) f(z) = g(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 \ldots$$

$$\implies f(z) = \frac{a_0}{(z - z_0)} + a_1 + a_2(z - z_0) + \ldots.$$

Hence $\text{Res}\left(f, z_0\right) = a_0 = g(z_0)$.

If z_0 is a pole of order m, then

$$(z - z_0)^m f(z) = g(z)$$

where $g(z) \neq 0$ on $D(z_0, r) \setminus \{z_0\}$.

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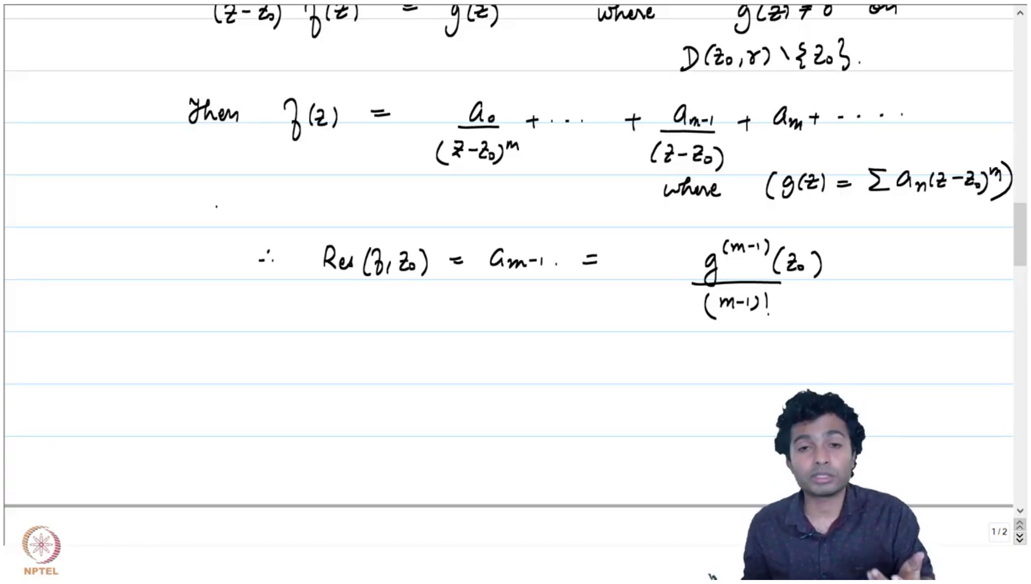

Hence we have

$$f(z) = \frac{a_0}{(z-z_0)^m} + \cdots + \frac{a_{m-1}}{(z-z_0)} + a_m + a_{m+1}(z-z_0) + \ldots$$

where $g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$. Therefore, $\operatorname{Res}(f, z_0) = a_{m-1} = \dfrac{g^{(m-1)}(z_0)}{(m-1)!}$.

THEOREM 1 (Residue Theorem). *Let Ω be an open subset of $\mathbb{C}$ and S be a finite subset of Ω. Suppose $f : \Omega \setminus S \longrightarrow \mathbb{C}$ is a bounded function. Let γ be a null homotopic closed curve in Ω. Then*

$$\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \sum_{j=1}^{k} W_\gamma(z_j) \operatorname{Res}(f, z_j)$$

where $S = \{z_1, z_2, \ldots, z_k\}$ and W_γ is the winding number.

PROOF. Define

$$g(z) = \sum_{j=1}^{k} \frac{\operatorname{Res}(f, z_j)}{(z-z_j)}.$$

Note that g is holomorphic on $\Omega \setminus S$.

Fix z_j and $r > 0$ be such that $\overline{D(z_j, r)} \subseteq \Omega$ and $D(z_j, r) \cap S = \{z_j\}$. We have

$$g(z) = \frac{\operatorname{Res}(f, z_j)}{(z-z_j)} + g_1(z)$$

where g_1 is the functions consists of the remaining terms of g. Then g_1 is holomorphic on $D(z_j, r)$. Note that $\text{Res}(g, z_j) = \text{Res}(f, z_j)$.

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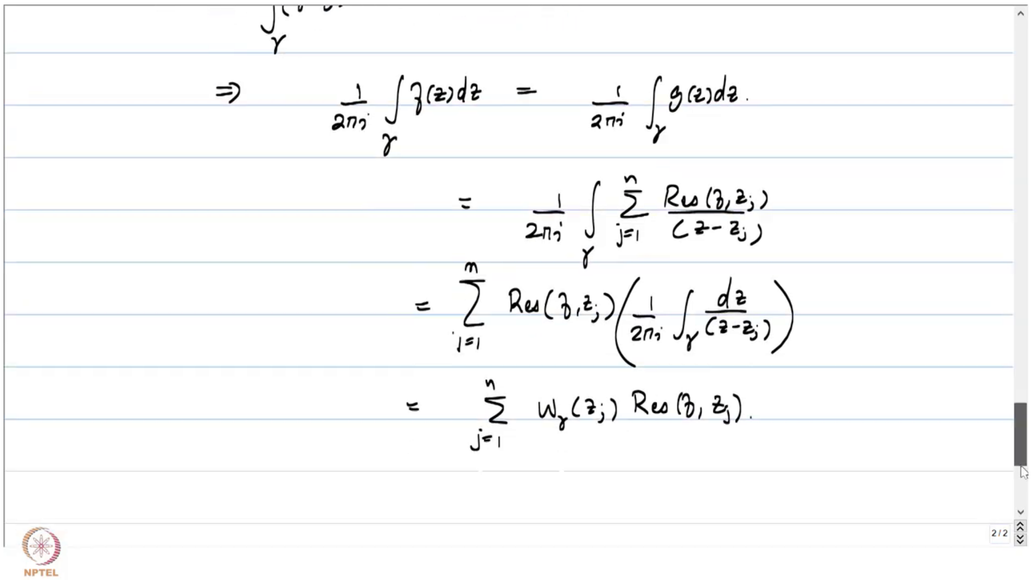

Define $F(z) = f(z) - g(z)$. If $\int_\gamma F(z)dz = 0$, then

$$\int_\gamma (f-g)(z)dz = 0$$
$$\implies \frac{1}{2\pi i}\int_\gamma f(z)dz = \frac{1}{2\pi i}\int_\gamma g(z)dz$$
$$= \frac{1}{2\pi i}\int_\gamma \sum_{j=1}^{k} \frac{\text{Res}(f, z_j)}{(z-z_j)}dz$$
$$= \sum_{j=1}^{k}\text{Res}(f, z_j)\left(\frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-z_j)}\right)$$
$$= \sum_{j=1}^{k} W_\gamma(z_j)\, Res(f, z_j).$$

Hence the proof will be completed if we manage to show that $\int_\gamma F(z)dz = 0$.

Let us try to figure out the Laurent series expansion of F. Since $F(z) = f(z) - g(z)$, then in $D(z_j, r) \setminus \{z_j\}$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_j)^n$$

and

$$g(z) = \frac{\operatorname{Res}(f, z_j)}{(z - z_j)} + \sum_{n=0}^{\infty} b_n (z - z_j)^n = \frac{a_{-1}}{(z - z_j)} + \sum_{n=0}^{\infty} b_n (z - z_j)^n$$

where the neighborhood $D(z_j, r)$ of z_j is same as above.

On $D(z_j, r) \setminus \{z_0\}$, we have

$$\begin{aligned} F(z) &= \sum_{n=-\infty}^{\infty} a_n (z - z_j)^n - \frac{a_{-1}}{(z - z_j)} + \sum_{n=0}^{\infty} b_n (z - z_j)^n \\ &= \sum_{n=2}^{\infty} a_{-n} (z - z_j)^{-n} + \sum_{n=0}^{\infty} a'_n (z - z_j)^n. \end{aligned}$$

Now, let us define a function G on $D(z_j, r) \setminus \{z_j\}$ by

$$G(z) = \sum_{n=2}^{\infty} \frac{a_{-n}(z - z_j)^{-n+1}}{-n+1} + \sum_{n=0}^{\infty} \frac{a'_n (z - z_j)^{n+1}}{(n+1)}.$$

Then, $G'(z) = F(z)$ and hence on $D(z_j, r) \setminus \{z_j\}$, F has an anti-derivative. Therefore, if C is a closed curve in $D(z_j, r) \setminus \{z_0\}$, we have

$$\int_C F(z)\,dz = 0. \tag{1}$$

If $z \in \Omega \setminus S$, then $r_1 > 0$ be such that $\overline{D(z, r_1)} \subseteq \Omega$ and $D(z, r_1) \cap S = \varnothing$. Then by Cauchy's theorem, for any closed curve C_1 in $D(z_0, r_1)$,

$$\int_{C_1} F(z)\,dz = 0. \tag{2}$$

Let $\gamma : [a, b] \longrightarrow \Omega \setminus S$ be null-homotopic in Ω. That is, there exists $H : [0,1] \times [a, b] \longrightarrow \Omega$ such that $\gamma_0 = \gamma$ and $\gamma_1 = \gamma_{z_0}$ and such that γ_s is closed curve for every $s \in [0,1]$, where $\gamma_{z_0}(t) = z_0$ for each $t \in [a, b]$.

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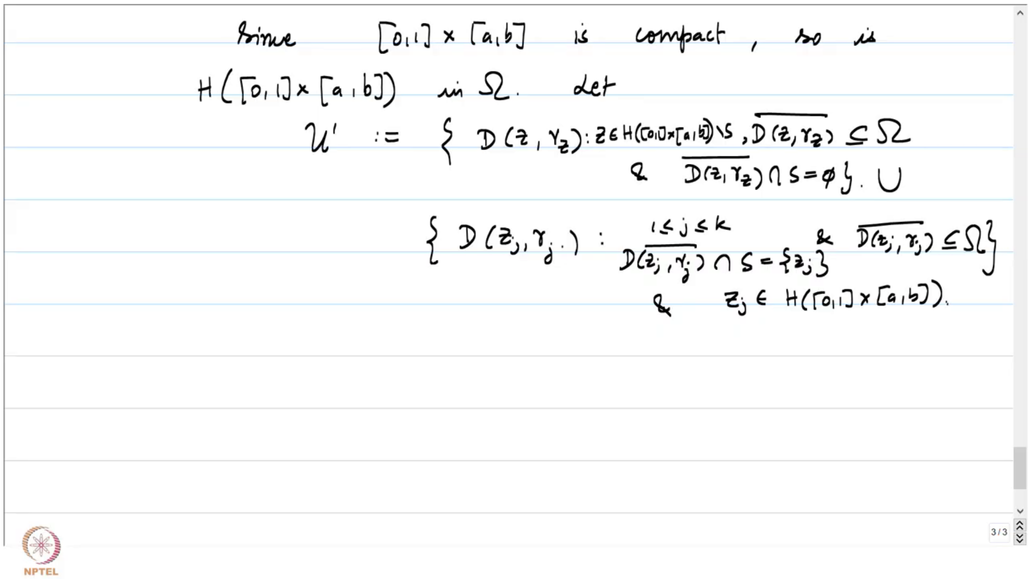

Since $[0,1] \times [a,b]$ is compact, so is $H([0,1] \times [a,b])$ in Ω. Let

$$\mathcal{U}' := \{D(z, r_z) : z \in H([0,1] \times [a,b]) \setminus S, \overline{D(z, r_z)} \subseteq \Omega \text{ and } \overline{D(z, r_z)} \cap S = \emptyset\} \bigcup$$
$$\{D(z_j, r_j) : 1 \le j \le k, z_j \in H([0,1] \times [a,b]), \overline{D(z_j, r_j)} \cap S = \{z_j\} \text{ and } \overline{D(z_j, r_j)} \subseteq \Omega\}.$$

Then $\mathcal{U}'$ will be an open cover of $H([0,1] \times [a,b])$. By compactness, let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a finite subcover and let ϵ be the Lebesgue number corresponding to $\mathcal{U}$.

By uniform continuity, there exists $\delta > 0$ such that

$$|H(s,t) - H(s',t')| < \frac{\epsilon}{4} \qquad \text{whenever } |s - s'| < \delta \text{ and } |t - t'| < \delta.$$

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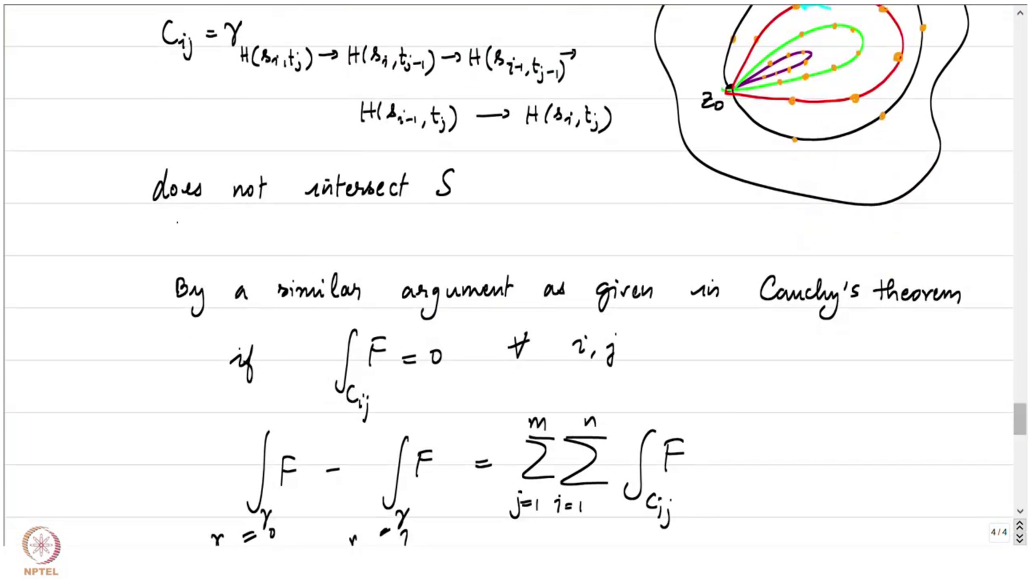

Consider the partitions $P_1 : 0 = a_0 < s_1 < \cdots < s_n = 1$ of $[0,1]$ and $P_2 : a = t_0 < t_1 < \cdots < t_m = b$ of $[a,b]$ such that the partition size is less than δ.

For every pair (i,j) such that $1 \le i \le n$ and $1 \le j \le m$,

$$C_{ij} = \gamma_{H(s_i,t_j)\to H(s_i,t_{j-1})\to H(s_{i-1},t_{j-1})\to H(s_i,t_j)}$$

does not intersect S.

By a similar argument as given in the Cauchy's theorem, if $int_{C_{i,j}} F(z)dz = 0$ then

$$\int_{\gamma=\gamma_0} F(z)dz = \int_{\gamma_{z_0}=\gamma_1} F(z) = \sum_{j=1}^{m}\sum_{i=1}^{n}\int_{C_{i,j}} F(z)dz. \tag{3}$$

Claim: $\int_{C_{i,j}} F(z)dz = 0$.

Note that $\left|C_{i,j}\right| < \epsilon$. Thus $\operatorname{diam}\left(C_{i,j}\right) < \epsilon$ and we have $\hat{C_{i,j}} \subset U$ for some $U \in \mathcal{U}$.

Hence by (1) and (2) and the fact that $C_{i,j} \cap S = \varnothing$, we have

$$\int_{C_{i,j}} F = 0.$$

Therefore by (3), we have

$$\int_{\gamma} F(z) = \int_{\gamma_{z_0}} F(z)dz.$$

□