Complex Analysis

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Lecture No – 36

Casorati-Weierstarss Theorem

We had explored the notions of removable singularity and the pole of a function f at an isolated singularity z_0 in great detail. We also studied the behaviour of the function as we approached these isolated singularities. For example, when we approach a removable singularity z_0 of f, we notice that the limit should exists and when we approach a pole of the function f at z_0, we noticed that the absolute value of the function blow up.

We also explored a Laurent series expansion of a function f defined on an annulus and we classified our singularities bases on how the negative coefficients of the Laurent series behave. However, we have still not really looked into what happens when a given isolated singularity is an essential singularity and how the function behaves as we approach an essential singularity.

Recall that $f(z) = e^{\frac{1}{1-z}}$ has essential singularity at $z_0 = 1$. Then, for

$$z_n = 1 - \frac{1}{2\pi i n}, \text{ we have } f(z_n) \longrightarrow 1 \text{ as } n \longrightarrow \infty$$

and for

$$z_n' = 1 - \frac{1}{2\pi i n + \frac{i\pi}{2}}, \text{ we have } f(z_n') \longrightarrow i \text{ as } n \longrightarrow \infty.$$

We will now prove a result which tells us that in fact the behaviour of essential singularities can get as bad as once can expect.

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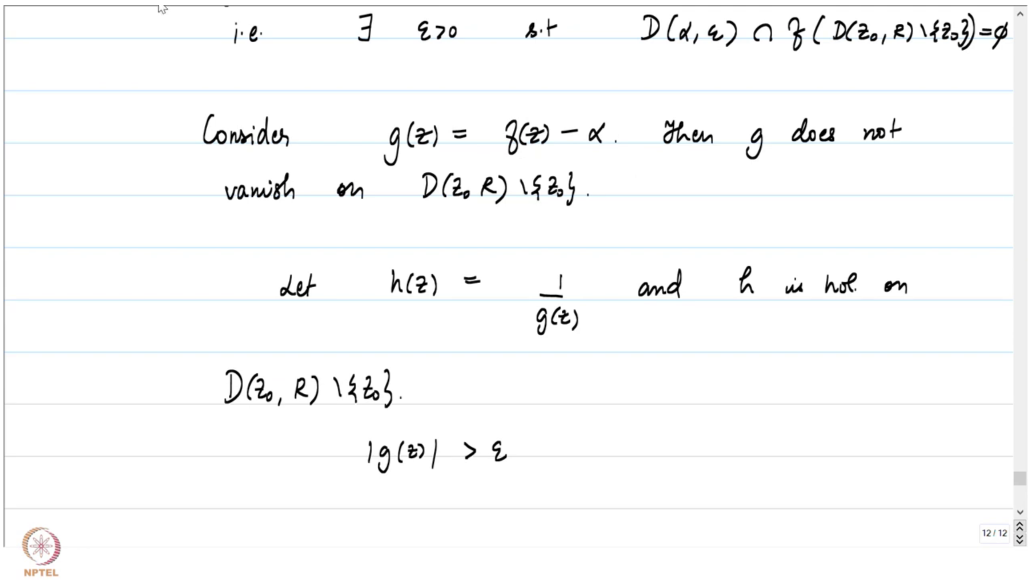

THEOREM 1 (Casorati-Weierstrass). *Let z_0 be an essential singularity of a function f. Then given $\alpha \in \mathbb{C}$, there exists a sequence $z_n \in D(z_0, R) \setminus \{z_0\}$ such that $z_n \longrightarrow z_0$ and $f(z_n) \longrightarrow \alpha$.*

PROOF. Suppose $\alpha \in \mathbb{C}$ be such that there does not exists a z_n such that $z_n \longrightarrow z_0$ and $f(z_n) \longrightarrow \alpha$. That is, there exists $\epsilon > 0$ such that $D(\alpha, \epsilon) \cap f(D(z_0, R) \setminus \{z_0\}) = \emptyset$.

Consider $g(z) = f(z) - \alpha$. Then g does not vanish on $D(z_0, R) \setminus \{z_0\}$. Let $h(z) = \frac{1}{g(z)}$ and h i holomorphic on $D(z_0, R) \setminus \{z_0\}$. Since $|g(z)| > \epsilon$, we have $h(z)| < \frac{1}{\epsilon}$ on $D(z_0, R) \setminus \{z_0\}$. Hence h is bounded on $D(z_0, R) \setminus \{z_0\}$. By the Riemann removable singularity theorem, z_0 is a removable singularity of h. Since h is not a constant function, on $D(z_0, R)$ we have

$$h(z) = (z - z_0)^m h_1(z)$$

where $m \geq 0$ and $h_1(z_0) \neq 0$. We may assume that $h_1(z) \neq 0$ on $D(z_0, R)$. Then we have

$$\frac{1}{f(z) - \alpha} = (z - z_0)^m h_1(z).$$

That is,

$$f(z) = \alpha + \frac{\frac{1}{h_1(z)}}{(z - z_0^m}.$$

Thus f has a pole of order m at z_0, which is a contradiction to the fact that z_0 is an essential singularity of f. Hence there exists a sequence $z_n \longrightarrow z_0$ such that $f(z_n) \longrightarrow \alpha$. □

Let us now define what is meant by a meromorphic function. But before we do that, let us recall the notion of the order of a pole of the function f at z_0.

Let f be a function with an isolated singularity at z_0. Let z_0 be a pole of order m at z_0. That is,

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

in $D(z_0, R) \setminus \{z_0\}$ where g is holomorphic on $D(z_0, R)$ and $g(z_0) \neq 0$.

We will similarly define the notion of order of zero of a holomorphic function f which vanishes at a point z_0. Suppose f be a non-constant holomorphic function in a neighborhood of a point z_0 such that $f(z_0) = 0$. Hence we have,

$$f(z) = (z - z_0)^m g(z)$$

where g is holomorphic and $g(z_0) \neq 0$. We then say that f has a zero of order m at z_0.

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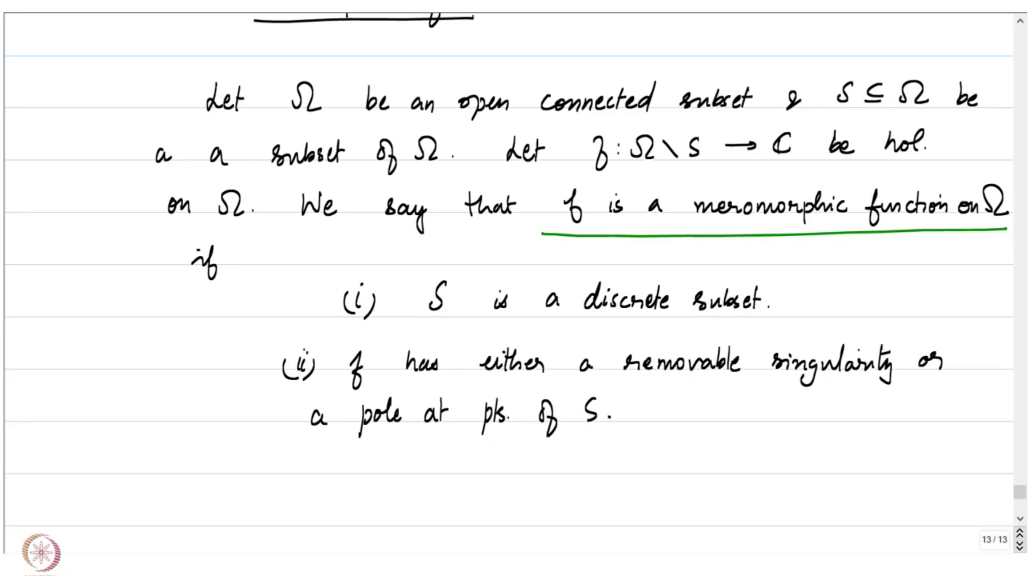

DEFINITION 1 (Meromorphic Function). Let Ω be an open connected subset and $S \subset \Omega$ be a subset of Ω. Let $f : \Omega \setminus S \longrightarrow \mathbb{C}$ be holomorphic on $\Omega \setminus S$. We say that f is a meromorphic function on Ω if

(1) S is a discrete set.
(2) f has either a removable singularity or a pole at points of S.

We say that two meromorphic functions f and g on Ω are equivalent if $f : \Omega \setminus S_1 \longrightarrow \mathbb{C}$ and $g : \Omega \setminus S_2 \longrightarrow \mathbb{C}$ satisfies $f(z) = g(z)$ on $\Omega \setminus (S_1 \cup S_2)$. It is left to the reader to check that if S_1 and S_2 are discrete sets, then so is $S_1 \cup S_2$.

Define $\mathscr{M}(\Omega) := \{$ Equivalence classes of meromorphic functions on $\Omega\}$.

Let $f, g \in \Omega$, i.e., $f : \Omega \setminus S_1 \longrightarrow \mathbb{C}$ and $g : \Omega \setminus S_1 \longrightarrow \mathbb{C}$ be such that $f = g$ on $\Omega \setminus (S_1 \cup S_2)$. Define $f + g$ to be the equivalence class of $(f + g) : \Omega \setminus (S_1 \cup S_2) \longrightarrow \mathbb{C}$ given by $(f + g)(z) = f(z) + g(z)$. Similarly define fg to be the equivalence class of $(fg) : \Omega \setminus (S_1 \cup S_2) \longrightarrow \mathbb{C}$ given by $(fg)(z) = f(z)g(z)$. Let $z_0 \in S_1 \cup S_2$. Then z_0 is either a removable singularity of fg or a pole of fg.

We know the behaviour of the functions f and g in a neighborhood of z_0,

$$f(z) = (z - z_0)^{m_1} f_1(z) \qquad \text{where } f_1(z_0) \neq 0,$$

$$g(z) = (z - z_0)^{m_2} g_1(z) \qquad \text{where } g_1(z_) \neq 0$$

and m_1, m_2 are non-negative or negative based on whether z_0 is a removable singularity or pole of f, g respectively. Then,

$$f(z)g(z) = (z - z_0)^{m_1 + m_2} f_1(z)g_1(z).$$

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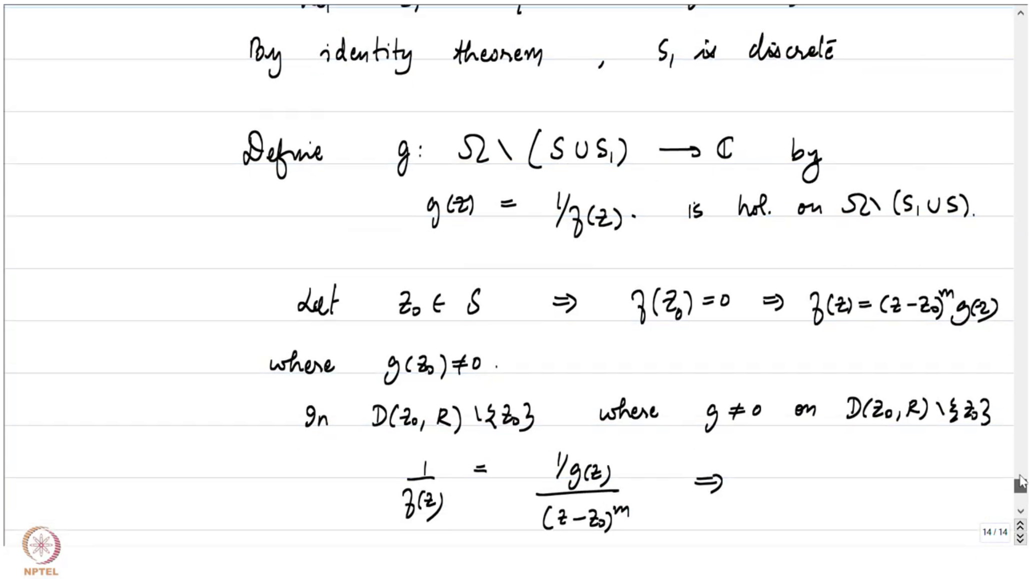

PROPOSITION 2. *The space of meromorphic function $\mathscr{M}(\Omega)$ on Ω is a field with operations defined above.*

PROOF. Let $f \in \mathscr{M}(\Omega)^*$, i.e., that is f is a non-zero meromorphic function on Ω. Then $f : \Omega \setminus S \longrightarrow \mathbb{C}$ is a non-zero holomorphic function. Let $S_1 = \{z \in \mathbb{C} : f(z) = 0\}$. By identity theorem, S_1 is discrete.

Define $g : \Omega \setminus (S \cup S_1) \longrightarrow \mathbb{C}$ by $g(z) = \dfrac{1}{f(z)}$. Then g is holomorphic on $\Omega \setminus (S \cup S_1)$. Let $z_0 \in S_1$. Then $f(z_0) = 0$. Thus we have $f(z) = (z - z_0)^{m_1} f_1(z_0)$, where $f_1(z_0) \neq 0$. Now, in $D(z_0, R) \setminus \{z_0\}$, we have $f_1(z) \neq 0$ for $z \in D(z_0, R) \setminus \{z_0\}$ and

$$g(z) = \frac{1}{f(z)} = \frac{\frac{1}{f_1(z)}}{(z - z_0)^{m_1}}.$$

That is, $g = \dfrac{1}{f}$ has a pole of order m_1 at z_0.

If $z_0 \in S$, then z_0 is either a pole or a removable singularity of f. First let us consider the case when z_0 is a pole of f.

If z_0 is a pole of f, then we have

$$f(z) = \frac{f_2(z)}{(z - z_0)^{m_2}}$$

where $f_2(z) \neq 0$ on $D(z_0, R) \setminus \{z_0\}$. Hence,

$$g(z) = \frac{1}{f(z)} = (z - z_0)^{m_2} \left(\frac{1}{f_2(z)} \right),$$

and $g = \dfrac{1}{f}$ has a removable singularity at z_0.

Now it is left as an exercise to the reader to check if z_0 is a removable singularity of f, then z_0 is either a removable singularity or a pole of g. Hence g is meromorphic on Ω and f is a unit in $\mathscr{M}(\Omega)$. □

DEFINITION 2 (Order of a meromorphic function at z_0). Let f be a meromorphic function on Ω. Then for $z_0 \in \Omega$, define the order of f at z_0 to be:

(i) if $z_0 \in S$ and z_0 is a removable singularity, then order of f at z_0 is the order of the zero at z_0 of f, i.e., if $f(z) = (z - z_0)^m g(z)$, then $ord_{z_0}(f) = m$. (Note that here m is non-negative.)

(ii) if $z_0 \in S$ and is a pole of order m, then $ord_{z_0}(f) = -m$.

(iii) if $z_0 \notin S$ and we have $f(z) = (z - z_0)^m g(z)$, where $m \geq 0$, then $ord_{z_0}(f) = m$.

(iv) if $f \equiv 0$, then $ord_{z_0}(f) = \infty$.

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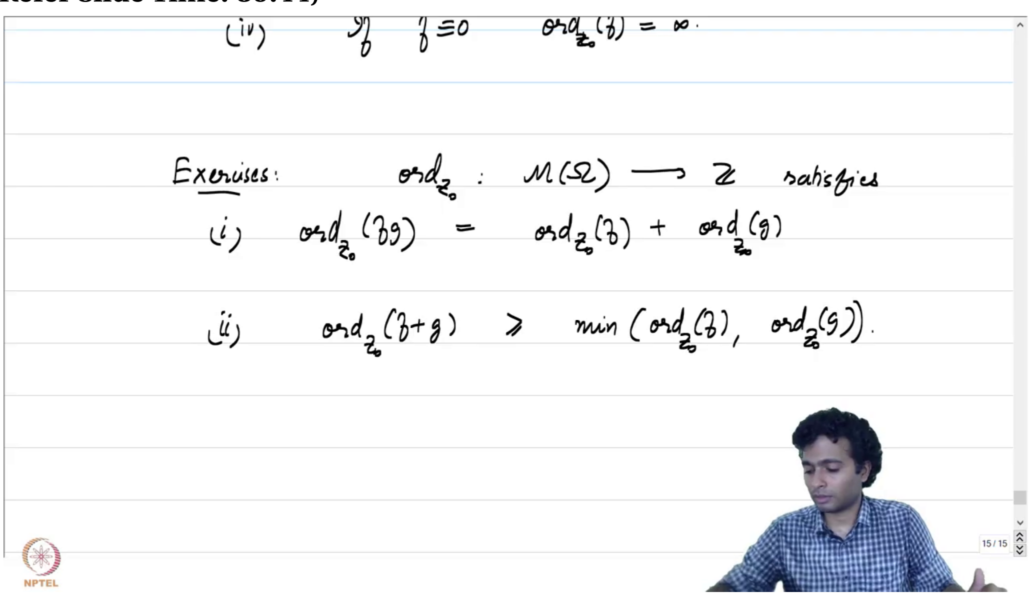

EXERCISE 3. Prove that $ord_{z_0} : \mathscr{M}(\Omega) \longrightarrow \mathbb{Z}$ satisfies

(1) $ord_{z_0}(fg) = ord_{z_0}(f) + ord_{z_0}(g)$

(2) $ord_{z_0}(f+g) \geq \min\big(ord_{z_0}(f), ord_{z_0}(g)\big)$.