Complex Analysis

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Lecture No – 35

Laurent series

In this lecture, our goal would be to generalize the expression that we gave for a function f which has a pole at z_0 and get hold of a more general such series expansion which is called the Laurent series expansion. In order to do that we will be now considering functions which are defined on an annulus and we will be considering doubly infinite series. So, before we really start developing our theory with Laurent series, let us set certain notations that we will be using.

DEFINITION 1 (Doubly Infinite Series). Let $\{z_n : n = 0, \pm 1, \pm 2, \ldots\}$ be a doubly infinite sequence of complex numbers. The expression $\sum_{n=-\infty}^{\infty} z_n$ is said to converge if $\sum_{n=0}^{\infty} z_n$ converges and $\sum_{n=1}^{\infty} z_{-n}$ converges.

We say that $\sum_{n=-\infty}^{\infty} z_n$ converges absolutely if $\sum_{n=0}^{\infty} z_n$ converges absolutely and $\sum_{n=1}^{\infty} z_{-n}$ converges absolutely.

Then we say that

$$\sum_{n=-\infty}^{\infty} z_n = \sum_{n=0}^{\infty} z_n + \sum_{n=1}^{\infty} z_{-n}$$

We define uniform convergence also very similarly. Let u_n, for $n = 0, \pm 1, \pm 2, \ldots,$ be functions defined on a set U. We say that $\sum_{n=-\infty}^{\infty} u_n$ converges uniformly on U if $\sum_{n=0}^{\infty} u_n(z)$ converges uniformly on U and $\sum_{n=1}^{\infty} u_{-n}(z)$ converges uniformly on U.

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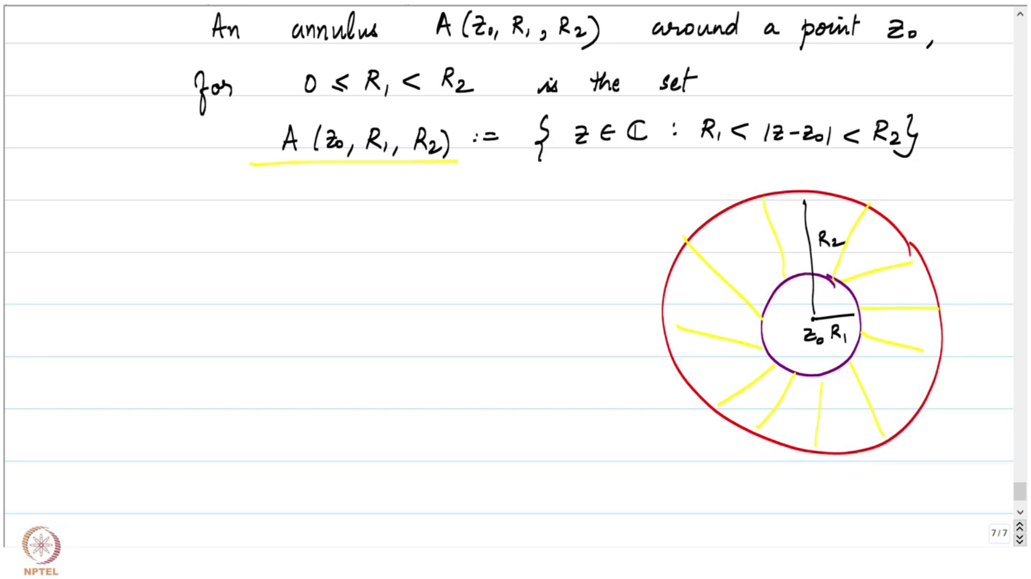

DEFINITION 2 (Annulus). An annulus $A(z_0, R_1, R_2)$ around a point $z_0 \in \mathbb{C}$, for $o \leq R_1 < R_2$, is the set

$$A(z_0, R_1, R_2) := \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}.$$

We now have all the ingredients to talk about the Laurent series development in an annulus.

THEOREM 1 (Laurent Series). *Let f be a function holomorphic on an annulus $A(z_0, R_1, R_2)$. Then there exist $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$ such that*

$$f(z) = \sum_{n=\infty}^{\infty} a_n (z - z_0)^n$$

called the Laurent series of f around z_0, where the doubly infinite series converges absolutely and uniform in $A(z_0, r_1, r_2)$ where $R_1 < r_1 < r_2 < R_2$. Moreover, if $\gamma(z) = z_0 + re^{it}$ for $t \in [0, 2\pi]$, where $R_1 < r < R_2$, then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

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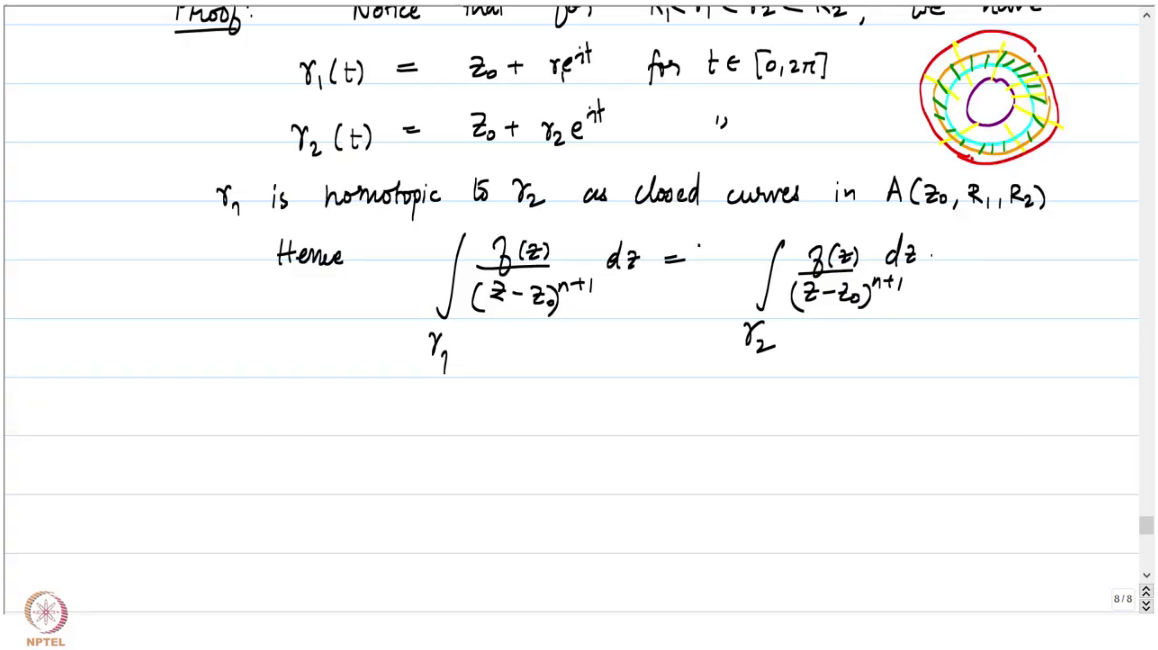

PROOF. For $R_1 < r_1 < r_2 < R_2$, let

$$\gamma_1(t) = z_0 + r_1 e^{it} \qquad \text{for } t \in [0, 2\pi]$$

and

$$\gamma_2(t) = z_0 + r_2 e^{it} \qquad \text{for } t \in [0, 2\pi].$$

Notice that γ_1 is homotopic to γ_2 as closed curves in $A(z_0, R_1, R_2)$.

Hence,

$$\int_{\gamma_1} \frac{f(z)}{(z-z_0)^{n+1}} dz = \int_{\gamma_2} \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

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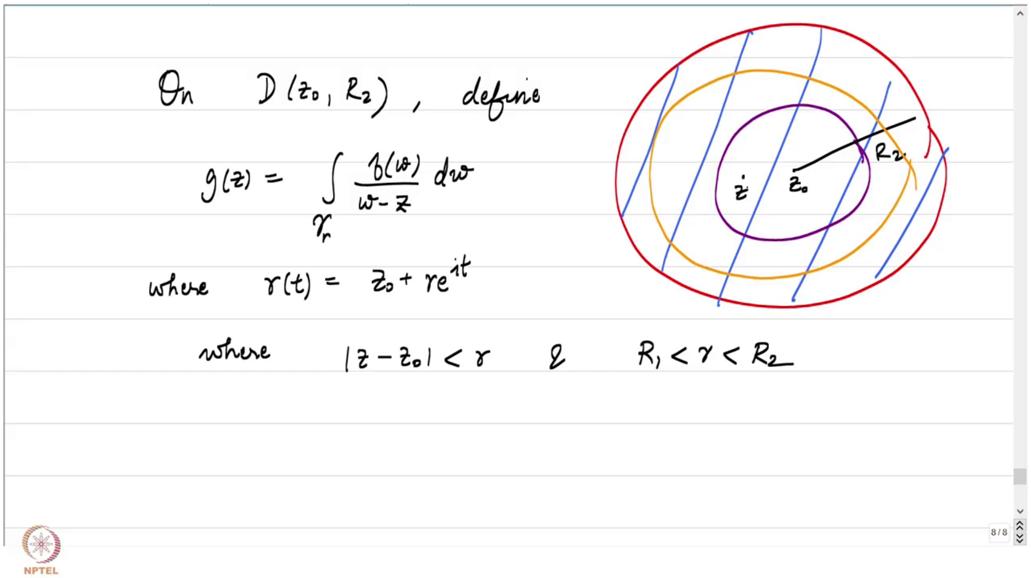

On $D(z_0, R_2)$, let $\gamma_r(t) = z_0 + re^{it}$. Then define

$$g(z) := \int_{\gamma_r} \frac{f(w)}{w-z} dz$$

where $|z - z_0| < r$ and $R_1 < r < R_2$.

Claim: g is continuous and holomorphic on $D(z_0, R_2)$.

Let $z' \in D(z_0, R_2)$. Let R be such that $R_1 < R < R_2$ and $|z' - z_0| < R$. Then, on $D(z', \epsilon)$ for $\epsilon > 0$ small, there exists $\delta > 0$ such that $|w - z| > \delta$, $|w - z'| > \delta$ for every $w \in \gamma_R$.

Then,

$$\begin{aligned} |g(z) - g(z')| &= \left| \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{w-z} dz - \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{w-z'} dz \right| \\ &= \left| \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)(z-z')}{(w-z)(w-z')} dw \right| \\ &\leq |z - z'| \left(\frac{M}{\delta^2} 2\pi R \right). \end{aligned}$$

Hence g is continuous at z'.

For $z \neq z'$,

$$\frac{g(z) - g(z')}{z - z'} = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{(w-z)(w-z')} dw.$$

Now it is left as an exercise to the reader to verify that

$$\lim_{\substack{z \to z' \\ z \neq z'}} \frac{g(z) - g(z')}{z - z'} = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{(w - z')^2} dw.$$

Hence g is holomorphic on $D(z_0, R_2)$.

Now, define a function h on $\{z : |z - z_0| > R_1\}$ by

$$h(z) = -\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w - z} dw$$

where $r \leq |z - z_0|$ and $R_1 < r < R_2$.

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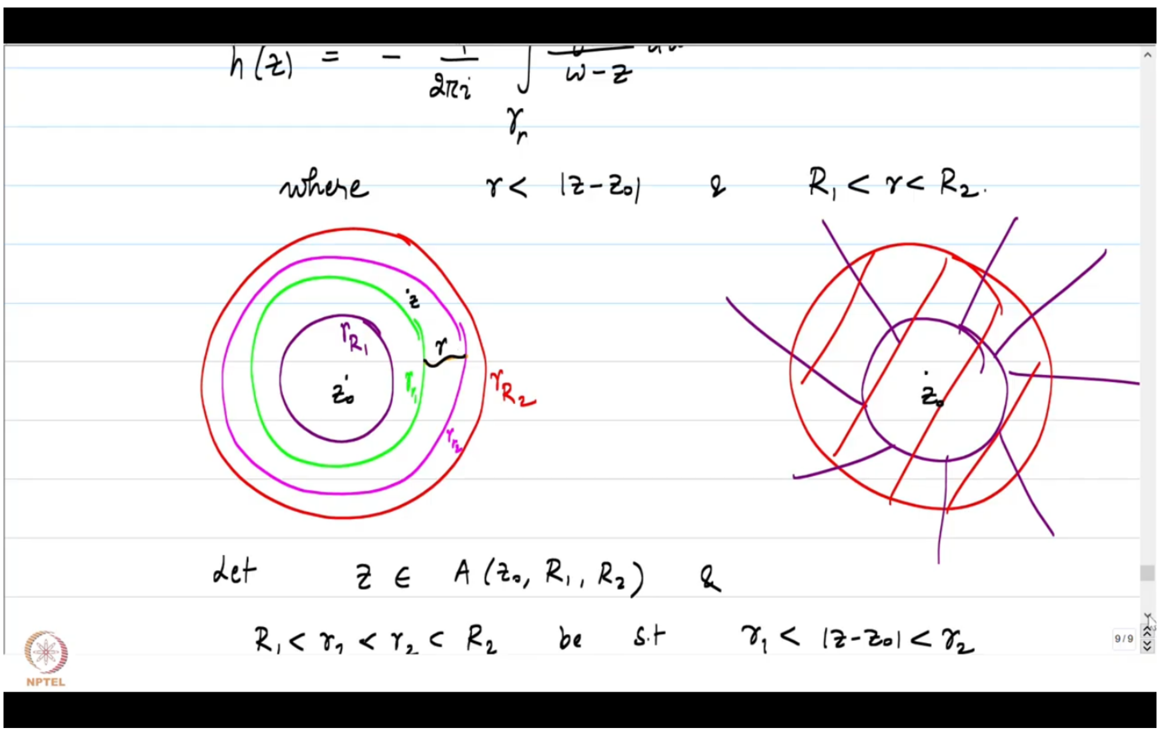

Let $z \in A(z_0, R_1, R_2)$ and $R_1 < r_1 < r_2 < R_2$ be such that $r_1 < |z - z_0| < r_2$. Consider a curve γ from $\gamma_{r_1}(0)$ to $\gamma_{r_2}(0)$ such that $z \notin \gamma$.

Let $\sigma = \gamma_{r_2} + (-\gamma) + (-\gamma_{r_1}) + \gamma$. Then σ is a closed curve in $A(z_0, R_1, R_2)$. Note that σ is null homotopic in $A(z_0, R_1, R_2)$.

Then, on $A(z_0, R_1, R_2)$, we have

$$\begin{aligned}
W_\sigma(z) f(z) &= \frac{1}{2\pi i}\int_\sigma \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\left(\int_{\gamma_{r_2}} \frac{f(w)}{w-z}dw + \int_{-\gamma} \frac{f(w)}{w-z}dw + \int_{(-\gamma_{r_1})} \frac{f(w)}{w-z}dw + \int_{\gamma} \frac{f(w)}{w-z}dw\right) \\
&= g(z) + h(z)
\end{aligned}$$

We know that $g(z)$ can be written as a power series expansion on $D(z_0, R_2)$. Let us see what happens to $h(z)$.

On $D(0, 1/R_1) \setminus \{0\}$, define

$$h_1(z) = h\left(z_0 + \frac{1}{z}\right).$$

Since

$$\left|z_0 + \frac{1}{z} - z_0\right| = \left|\frac{1}{z}\right| > R_1,$$

h_1 is well defined as $\{z : |z - z_0| > R_1\}$ is the domain of definition of h.

Claim: h_1 is locally bounded around 0.

For $|z| > s$, where $s > 0$ is some large real,

$$\begin{aligned}
|h(z)| &= \left|-\frac{1}{2\pi i}\int_{\gamma_{r_1}} \frac{f(w)}{w-z}dw\right| \\
&\le \frac{M 2\pi r_1}{2\pi d(z, \gamma_{r_1})} \\
&= \frac{M r_1}{d(z, \gamma_{r_1})} \\
&< 1.
\end{aligned}$$

Since $h_1(z) = h\left(z_0 + \frac{1}{z}\right)$ where $|z| < \epsilon$ for $\epsilon > 0$ small, we have $\left|z_0 + \frac{1}{z}\right| > s$ and $|h_1(z)| = \left|h\left(z_0 + \frac{1}{z}\right)\right| < 1$. Hence h_1 has a removable singularity at 0.

That is, we have, $h_1(z) = \sum_{n=0}^{\infty} b_n z^n$. Since $h_1(0) = 0$, we can rewrite the series expansion for h_1 on $D(0, 1/R_1)$ as

$$h_1(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Let $z_0 + \frac{1}{z} = w$. Then $z = \frac{1}{w-z_0}$. Hence $h(w) = h_1\left(\frac{1}{w-z_0}\right)$.

Thus, if $z \in \{z \in \mathbb{C} : |z - z_0| > R_1\}$,

$$\begin{aligned}
h(z) &= h_1\left(\frac{1}{z-z_0}\right) \\
&= \sum_{n=1}^{\infty} b_n \left(\frac{1}{z-z_0}\right)^n \\
&= \sum_{n=-1}^{-\infty} a_n (z-z_0)^n.
\end{aligned}$$

Thus,

$$\begin{aligned}
f(z) &= g(z) + h(z) \\
&= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n} \\
&= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n.
\end{aligned}$$

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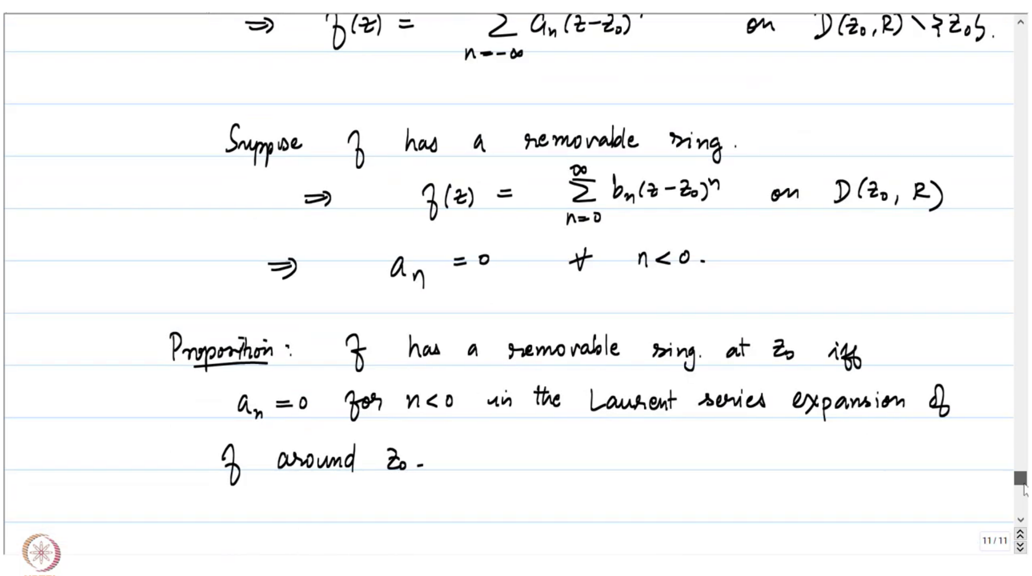

Suppose f has an isolated singularity at z_0. Then, for some $R > 0$, we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad \text{on } D(z_0, R) \setminus \{z_0\}.$$

Suppose z_0 is a removable singularity of f, then we have

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n \qquad \text{on } D(z_0, R).$$

Hence $a_n = 0$ for every $n < 0$.

The converse of above observation is also true. This can be proved if one back track what we did above. That is if $a_n = 0$ for every $n < 0$, then f has a power series expansion around z_0. Hence z_0 is a removable singularity. These observation will the following proposition.

PROPOSITION 2. *Let z_0 be an isolated singularity of the function f. Then f has a removable singularity at z_0 if and only if $a_n = 0$ for $n < 0$ in the Laurent series expansion of f around z_0.*

If f has a pole at z_0 of order m, then $(z - z_0)^m f$ has a removable singularity at z_0. Thus, on $D(z_0, R) \setminus \{z_0\}$, we have,

$$f(z) = \frac{b_{-m}}{(z - z_0)^m} + \cdots + \frac{b_{-1}}{(z - z_0)} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Hence $a_n = 0$ for each $n < -m$ in the Laurent series expansion around z_0 of f.

Here also, the converse of the above observation is true. That is, if $a_n = 0$ for $n < -m$ in the Laurent series expansion around z_0 of f, then f has a pole of order m at z_0. The proof is trivial, as it follows from the definition.

PROPOSITION 3. *Let z_0 be an isolated singularity of the function f. Then f has a pole of order m at z_0 if and only if $a_n = 0$ for $n < -m$ in the Laurent series expansion of f in $D(z_0, R) \setminus \{z_0\} = A(z_0, 0, R)$.*

PROPOSITION 4. *Let z_0 be an isolated singularity of the function f. Then f has an essential singularity at z_0 if and only if $a_n \neq 0$ for infinitely many negative integers n.*