Complex Analysis

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Lecture No – 34

Pole of a Function

In the last lecture we saw what was meant by a removable singularity of a function f. We observed that an isolated singularity of f is a removable singularity if and only if f is locally bounded around that point. We also noted that if an isolated singularity z_0 is a removable singularity then the limit $\lim_{z\to z_0} f(z)$ should exist. By considering the behavior of $f(z)$ as z goes to the singularity, let us now study the other types of singularities that can occur.

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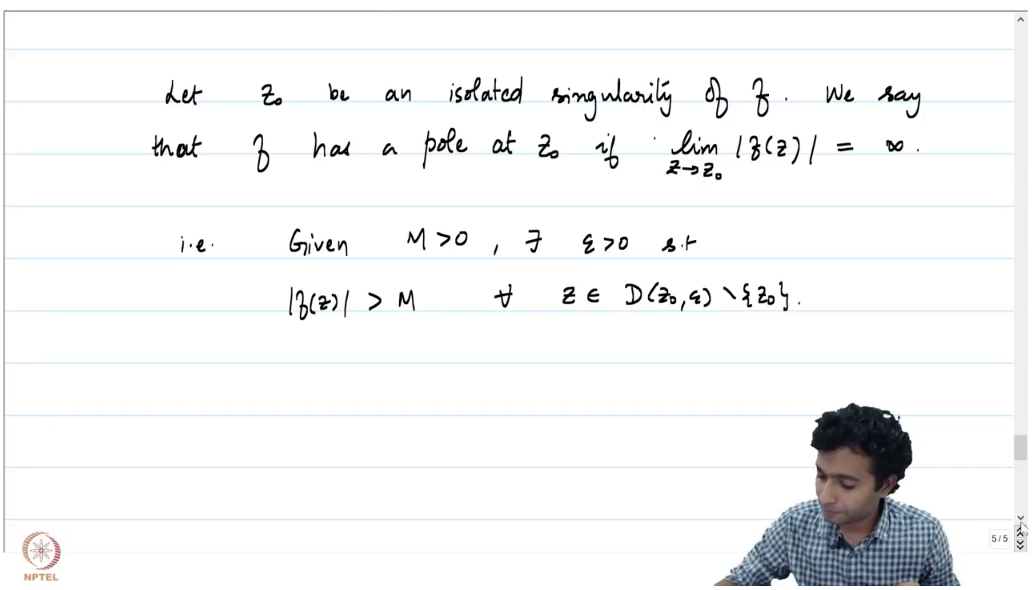

DEFINITION 1 (Pole of a function). Let z_0 be an isolated singularity of f. We say that f has a pole at z_0 if $\lim_{z\to z_0} |f(z)| = \infty$. That is, given $M > 0$, there exists $\epsilon > 0$ such that $|f(z)| > M$ for each $z \in D(z_0, \epsilon) \setminus \{z_0\}$.

EXAMPLE 1.

- Let $f(z) = \dfrac{\cos(z)}{z}$. Then 0 is an isolated singularity of f. Also, $\lim_{z\to 0}\left|\dfrac{\cos(z)}{z}\right| = \infty$. Hence f has a pole at $z_0 = 0$.
- Recall that if $f(z) = e^{\frac{1}{1-z}}$, then 1 is an isolated singularity of f and $\lim_{z\to 1} f(z)$ does not exists. Hence f does not have a pole at 1.

DEFINITION 2 (Essential Singularity of a function). Let z_0 be an isolated singularity of f. Then z_0 is called an essential singularity of f if it is neither a removable singularity nor a pole of f.

Let z_0 be a pole of f. Then $\lim_{z\to z_0} |f(z)| = \infty$. Hence there exists $R > 0$ such that on $D(z_0, R) \setminus \{z_0\}$, $f(z) \neq 0$. In particular, on $D(z_0, R) \setminus \{z_0\}$, $g(z) := \dfrac{1}{f(z)}$ is holomorphic. Then z_0 is an isolated singularity of g.

Let $R > 0$ be such that $|f(z)| > M$ on $D(z_0, R) \setminus \{z_0\}$. That is,on $D(z_0, R) \setminus \{z_0\}$, we have $|g(z)| < \dfrac{1}{M}$. By Riemann removable singularity theorem, z_0 is a removable singularity of g. Now it is left as an exercise for the reader to verify that $\lim_{z\to z_0} g(z) = 0$.

Hence let us define h to be

$$h(z) := \begin{cases} g(z) & ,\text{on } D(z_0,R) \setminus \{z_0\} \\ 0 & ,\text{for } z = z_0. \end{cases}$$

Then h is holomorphic. By factorization and principle of analytic continuation, on $D(z_0, R)$,

$$h(z) = (z - z_0)^m h_1(z) \qquad \text{where } h_1(z) \neq 0 \text{ and } m \geq 1.$$

Hence on $D(z_0, R) \setminus \{z_0\}$, we have

$$\frac{1}{f(z)} = g(z) = (z - z_0)^m h_1(z).$$

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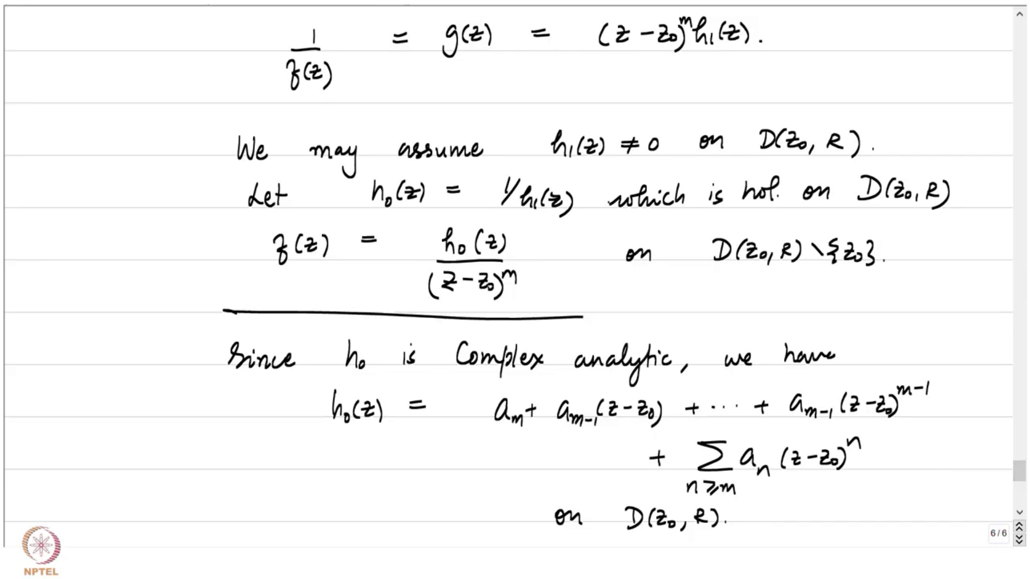

$$\frac{1}{f(z)} = g(z) = (z-z_0)^m h_1(z).$$

We may assume $h_1(z) \neq 0$ on $D(z_0, R)$.
Let $h_0(z) = 1/h_1(z)$ which is hol. on $D(z_0, R)$

$$f(z) = \frac{h_0(z)}{(z-z_0)^m} \quad \text{on } D(z_0,R) \setminus \{z_0\}.$$

Since h_0 is Complex analytic, we have

$$h_0(z) = a_m + a_{m-1}(z-z_0) + \cdots + a_{m-1}(z-z_0)^{m-1} + \sum_{n \geq m} a_n (z-z_0)^n$$

on $D(z_0, R)$.

Since $h_1(z) \neq 0$ on $D(z_0, R)$, let $h_0(z) = \dfrac{1}{h_1(z)}$, which is holomorphic on $D(z_0, R)$. Thus, we have, on $D(z_0, R) \setminus \{z_0\}$,

$$f(z) = \frac{h_0(z)}{(z-z_0)^m}.$$

Since h_0 is complex analytic, on $D(z_0, R)$, we have

$$h_0(z) = a_0 + a_1(z-z_0) + \cdots + a_{m-1}(z-z_0)^{m-1} + \sum_{n \geq m} a_n (z-z_0)^n \qquad \text{where } a_0 \neq 0.$$

Hence, on $D(z_0, R) \setminus \{z_0\}$,

$$f(z) = \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots + \frac{a_{m-1}}{(z-z_0)} + \sum_{n \geq m} a_n (z-z_0)^{n-m}, \; a_0 \neq 0.$$

Put $g_1(z) = \sum_{n \geq m} a_n (z-z_0)^{n-m}$. Observe that g_1 is a holomorphic function on $D(z_0, R)$. The function S given by

$$S(z) = \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots + \frac{a_{m-1}}{(z-z_0)}$$

is called the singular part of f around the pole z_0 and the integer m is called the order of the pole at z_0.