Complex Analysis

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Lecture No – 33

Singularities of a Holomorphic Function

Often times we are interested in studying functions f which are defined and holomorphic on a set $\Omega \setminus S$ where the set S is called the set of singularities of f. Here we will be discussing the notion of singularities of a holomorphic function. Singularities come in varying levels of severity and we will be exploring them one by one. Let us begin by defining what is meant by a singularity of a holomorphic function.

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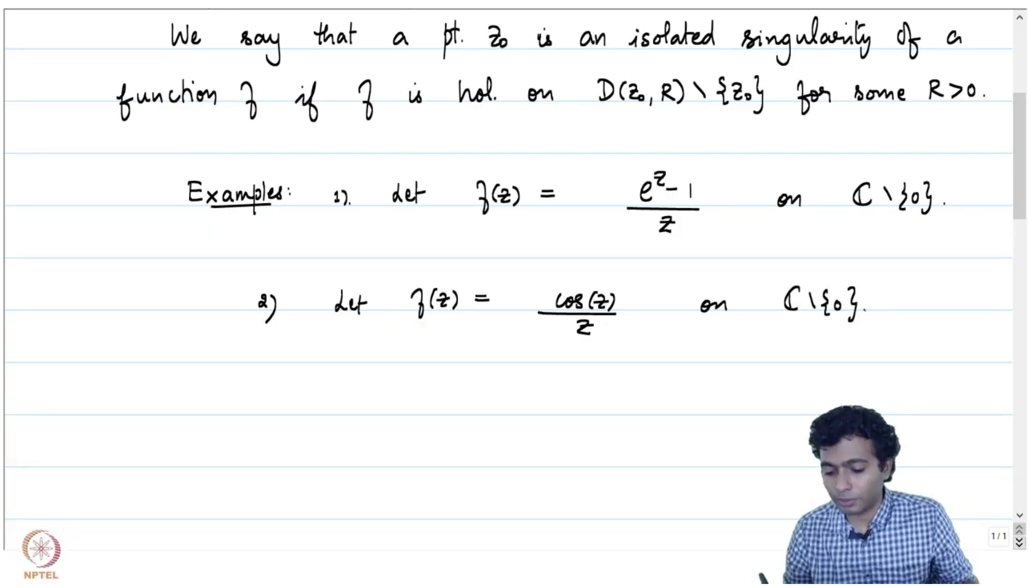

Definition 1 (Isolated Singularity). We say that a point z_0 is an isolated singularity of a function f if f is holomorphic on $D(z_0, R) \setminus \{z_0\}$ for some $R > 0$.

Example 1.

(1) Let $f(z) = \dfrac{e^z - 1}{z}$ on $\mathbb{C} \setminus \{0\}$.

(2) Let $g(z) = \dfrac{\cos(z)}{z}$ on $\mathbb{C} \setminus \{0\}$.

(3) Let $h(z) = e^{\dfrac{1}{1-z}}$ on $\mathbb{C} \setminus \{1\}$.

Here the functions f and g have isolated singularity at $z_0 = 0$ and the function h has an isolated singularity at $z_0 = 1$.

DEFINITION 2 (Removable Singularity). Let z_0 be an isolated singularity of a holomorphic function f, i.e., there exists an $R > 0$ such that f is holomorphic on $D(z_0, R) \setminus \{z_0\}$. We say that z_0 is removable singularity of f if there exists a function g holomorphic on $D(z_0, R)$ and such that $f(z) = g(z)$ on $D(z_0, R) \setminus \{z_0\}$.

Let us revisit the examples.

(1) Let $f(z) = \dfrac{e^z - 1}{z}$. Then f has an isolated singularity at $z_0 = 0$. Let $R > 0$ be any real number. Note that in $D(0, R)$, we have

$$e^z - 1 = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!}.$$

Define $F(z) := \sum_{n=1}^{\infty} \dfrac{z^{n-1}}{n!}$. Then F is holomorphic on $D(0, R)$. (Why?)

Hence on $D(0, R)$, we have

$$zF(z) = z\left(\sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}\right) = \sum_{n=1}^{\infty} \frac{z^n}{n!}.$$

Hence $zF(z) = e^z - 1$. For $z \neq 0$, in particular on $D(0, R) \setminus \{0\}$, we have

$$F(z) = \frac{e^z - 1}{z} = f(z).$$

Note that if an isolated singularity z_0 of f is a removable singularity, then $\lim\limits_{\substack{z \to z_0 \\ z \neq z_0}} f(z)$ exists.

(2) Let $g(z) = \dfrac{\cos(z)}{z}$. Then g has an isolated singularity at 0. Note that $\cos(0) = 1$. We know that $\cos(z)$ is a continuous function and hence there exists $R > 0$ such

that on $D(0,R)$, $|\cos(z)| < M$ for some $M > 0$. Observe that,

$$\lim_{z\to 0}\left|\frac{\cos(z)}{z}\right| = \infty.$$

If 0 were to be a removable singularity, $\lim_{z\to 0}\left|\frac{\cos(z)}{z}\right|$ should exists. Hence 0 is not a removable singularity.

(3) Let $h(z) = e^{\frac{1}{1-z}}$. Then h has an isolated singularity at 1.

Let $z_n = 1 - \frac{1}{2\pi i n}$, so that $z_n \longrightarrow 1$ as $n \longrightarrow \infty$. Then

$$h(z_n) = e^{\frac{1}{1-\left(1-\frac{1}{2\pi i n}\right)}} = e^{2\pi i n} = 1.$$

Hence $\lim_{n\to\infty} h(z_n) = 1$.

Let $z_n' = 1 - \frac{1}{2\pi i n + \frac{\pi i}{2}}$. Then $z_n' \longrightarrow 1$ as $n \longrightarrow \infty$. Now,

$$h(z_n') = \exp\left(\frac{1}{1-\left(1-\frac{1}{2\pi i n+\frac{\pi i}{2}}\right)}\right) = \exp\left(2\pi i n + \frac{\pi i}{2}\right) = i.$$

Here, $\lim_{n\to\infty} h(z_n') = i$. Hence h does not have a removable singularity at $z = 1$.

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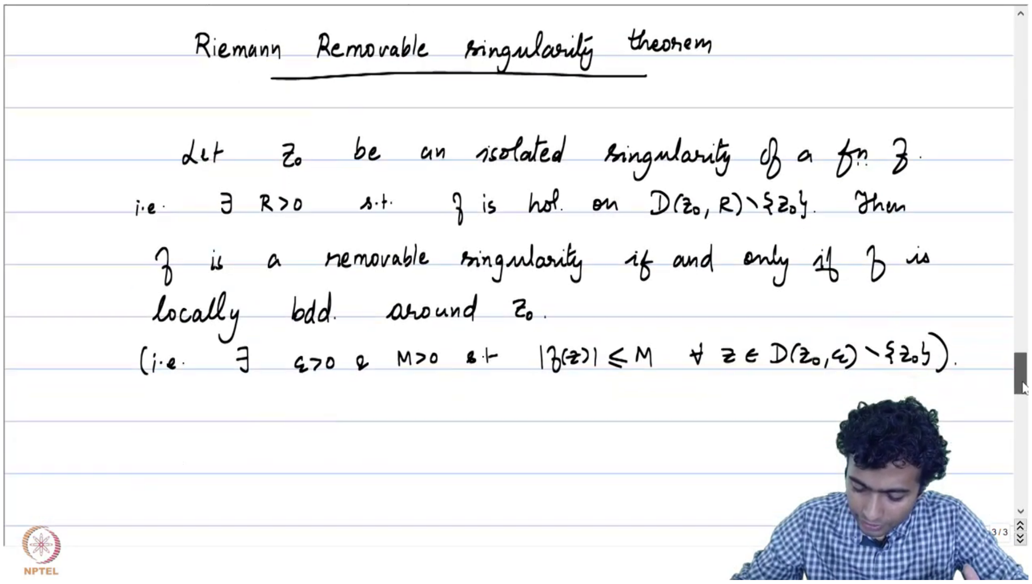

THEOREM 2 (Riemann Removable Singularity Theorem). *Let z_0 be an isolated singularity of a function f, i.e., there exists $R > 0$ such that f is holomorphic on $D(z_0, R) \setminus \{z_0\}$. Then f has a removable singularity at z_0 if and only if f is locally bounded around z_0 (i.e. there exist $\epsilon > 0$ and $M > 0$ such that $|f(z)| \leq M$ for each $z \in D(z_0, \epsilon) \setminus \{z_0\}$).*

PROOF. ($\Rightarrow$) Assume that f has a removable singularity at z_0. Then there exists a holomorphic function g on $D(z_0, R)$ that agrees with f on $D(z_0, R) \setminus \{z_0\}$ and since g is continuous at z_0, g is bounded in a neighborhood $D(z_0, \epsilon)$ of z_0. Hence f is bounded in $D(z_0, \epsilon) \setminus \{z_0\}$.

($\Leftarrow$) Let us now assume that f is locally bounded in a neighborhood $D(z_0, R) \setminus \{z_0\}$.

Define $h : D(z_0, R) \longrightarrow \mathbb{C}$ by

$$h(z) := \begin{cases} (z - z_0) f(z), & z \in D(z_0, R) \setminus \{z_0\} \\ 0, & z = z_0. \end{cases}$$

Then h is continuous at z_0 (verify). If h was holomorphic on $D(z_0, R)$, then there exists a power series expansion of h on $D(z_0, R)$ given by

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and since $h(z_0) = 0$, we have

$$h(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

That is, on $D(z_0, R)$ we have,

$$h(z) = (z - z_0)\left(\sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}\right)$$

Define $g(z) := \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}$. Then on $D(z_0, R) \setminus \{z_0\}$, we have

$$h(z) = (z - z_0) f(z) = (z - z_0) g(z) \implies f(z) = g(z).$$

Hence f has a removable singularity at z_0.

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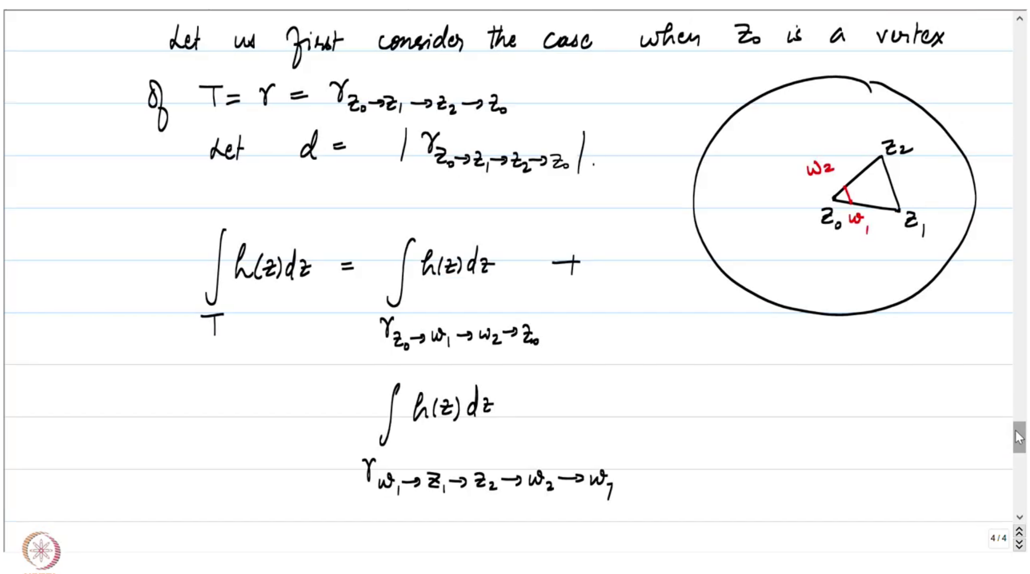

Claim: h is holomorphic on $D(z_0, R)$.

We shall use Morera's theorem for proving above claim.

If a triangular curve T satisfies $z_0 \notin \hat{T}$, then T is hull homotopic and we have $\int_T h = 0$.

Let us consider the case when z_0 is a vertex of $T = \gamma_{z_0 \to z_1 \to z_2 \to z_0}$, where $z_1, z_2 \in D(z_0, R)$. Let $d = \left|\gamma_{z_0 \to z_1 \to z_2 \to z_0}\right|$. Now if $w_1 \in \gamma_{z_0 \to z_1}$ and $w_2 \in \gamma_{z_2 \to z_0}$, then by Cauchy's theorem we have

$$\int_{\gamma_{w_1\to z_1\to z_2\to w_2\to w_1}} h(z)dz = 0.$$

Hence,

$$\begin{aligned}\int_T h(z)dz &= \int_{\gamma_{z_0\to w_1\to w_2\to z_0}} h(z)dz + \int_{\gamma_{w_1\to z_1\to z_2\to w_2\to w_1}} h(z)dz \\ &= \int_{\gamma_{z_0\to w_1\to w_2\to z_0}} h(z)dz.\end{aligned}$$

Since h is continuous, given $\epsilon > 0$, let $\delta > 0$ be such that $|h(w) - h(z_0)| < \dfrac{\epsilon}{d}$, whenever $|w - z_0| < \delta$. But $h(z_0) = 0$, then $|h(w)| < \dfrac{\epsilon}{d}$, whenever $|w - z_0| < \delta$.

Now pick w_1, w_2 above such that $\hat{\gamma}_{z_0\to w_1\to w_2\to z_0} \subseteq D(z_0,\delta)$. Then,

$$\left|\int_T h(z)dz\right| = \left|\int_{\gamma_{z_0\to w_1\to w_2\to z_0}} h(z)dz\right| \le \frac{\epsilon}{d}\left|\gamma_{z_0\to w_1\to w_2\to z_0}\right| = \epsilon.$$

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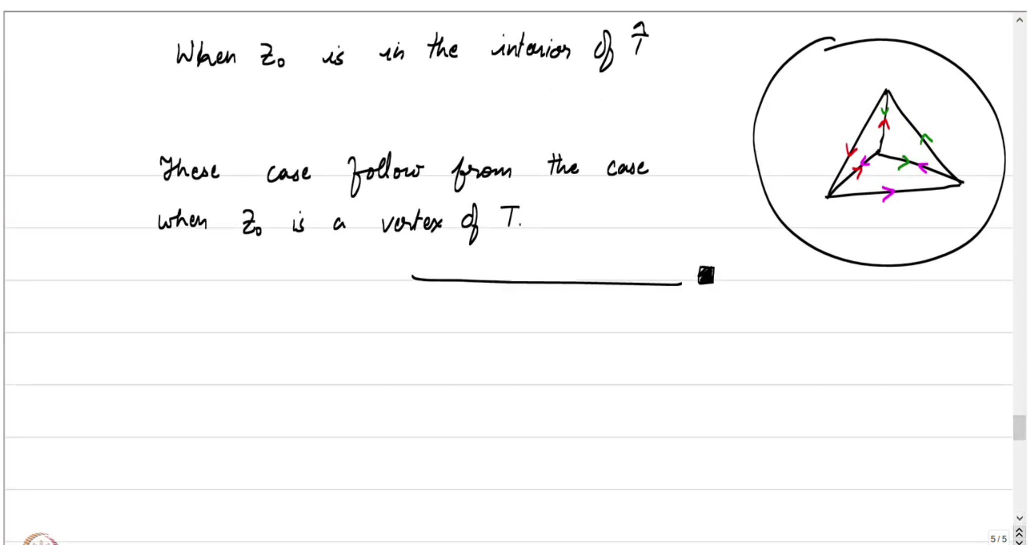

Hence $\left|\int_T h(z)dz\right| < \epsilon$. Since the ϵ was arbitrary, we have $\int_T h(z)dz = 0$.

Now let us consider the case where z_0 is an edge of T. If $T = \gamma_{z_1 \to z_2 \to z_3 \to z_1}$ and without loss of generality, let $z_0 \in \gamma_{z_1 \to z_2}$. Then,

$$\int_T h(z)dz = \int_{\gamma_{z_1 \to z_0 \to z_3 \to z_1}} h(z)dz + \int_{\gamma_{z_2 \to z_3 \to z_0 \to z_2}} h(z)dz.$$

But we have already proved the case where z_0 is a vertex of the triangle. Hence,

$$\int_{\gamma_{z_1 \to z_0 \to z_3 \to z_1}} h(z)dz = 0, \qquad \int_{\gamma_{z_2 \to z_3 \to z_0 \to z_2}} h(z)dz = 0.$$

Therefore,

$$\int_T h(z) = 0.$$

Finally, if z_0 is in the interior of $\hat{T}$, where $T = \gamma_{z_1 \to z_2 \to z_3 \to z_1}$. Here also we can apply the case where z_0 is a vertex by decomposing the integral $\int_T h(z)dz$ as

$$\int_T h(z)dz = \int_{\gamma_{z_1 \to z_0 \to z_3 \to z_1}} h(z)dz + \int_{\gamma_{z_2 \to z_0 \to z_1 \to z_2}} h(z)dz + \int_{\gamma_{z_3 \to z_0 \to z_2 \to z_3}} h(z)dz.$$

But each of the integral in the RHS is 0, hence $\displaystyle\int_T h(z)dz = 0.$ $\square$