Complex Analysis

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Lecture No – 32

Problem Session

PROBLEM 1. Let p be a polynomial of degree n with complex coefficients. Suppose the roots of the polynomial p are contained in $D(0,R)$ for large R. Then prove that

$$\frac{1}{2\pi i}\int_{\gamma}\frac{p'(z)}{p(z)}dz = n$$

where $\gamma(t) = Re^{it}$ for $t \in [0,2\pi]$.

SOLUTION 1. By fundamental theorem of algebra, we have

$$p(z) = a_n(z-z_1)\cdots(z-z_n).$$

Now,

$$p'(z) = \sum_{j=1}^{n} a_n(z-z_1)\ldots\widehat{(z-z_j)}\ldots(z-z_n),$$

where $\widehat{(z-z_j)}$ means $(z-z_j)$ does not appear in the expression.

Away from $z_1,\ldots,z_n$, we have the function,

$$\begin{aligned}\frac{p'(z)}{p(z)} &= \frac{\sum_{j=1}^{n} a_n(z-z_1)\ldots\widehat{(z-z_j)}\ldots(z-z_n)}{a_n(z-z_1)\cdots(z-z_n)}\\ &= \sum_{j=1}^{n}\frac{1}{z-z_j}.\end{aligned}$$

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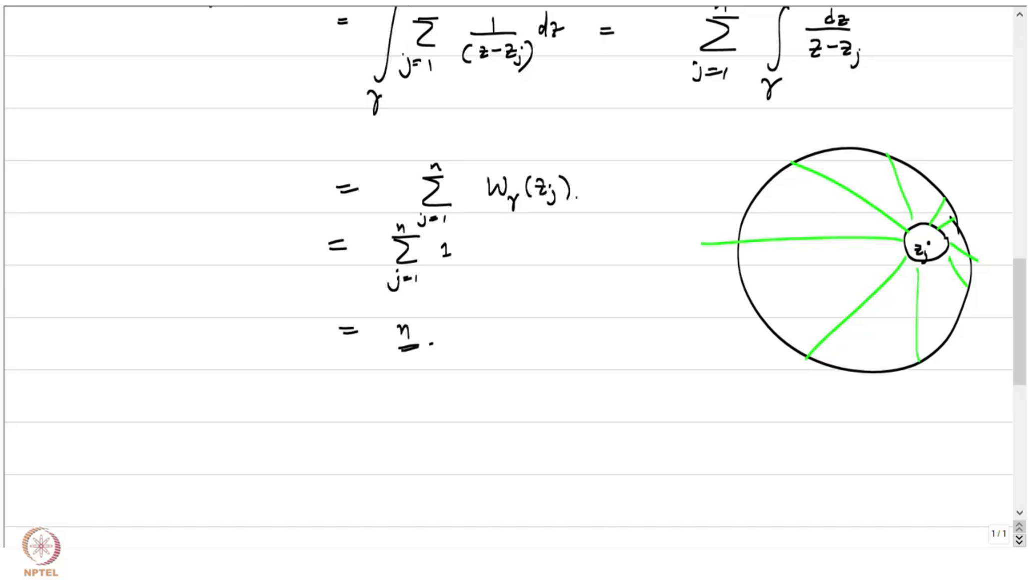

Hence,

$$\begin{aligned}\int_\gamma \frac{p'(z)}{p(z)}dz &= \int_\gamma \sum_{j=1}^{n} \frac{1}{z-z_j}dz \\ &= \sum_{j=1}^{n}\int_\gamma \frac{dz}{z-z_j} \\ &= \sum_{j=1}^{n} W_\gamma(z_j) \\ &= n.\end{aligned}$$

PROBLEM 2. Let $f:\Omega \longrightarrow \mathbb{C}$ be a non-constant holomorphic function defined on an open set Ω. Suppose $a_1, a_2, \ldots, a_n$ be points in Ω such that $f(a_i) = \alpha$ for some $\alpha \in \mathbb{C}$. Let γ be a closed continuous differentiable curve which is null homotopic and such that $a_j \notin \gamma$. Then prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}dz = \sum_{j=1}^{n} m_j W_\gamma(a_j)$$

where m_j are positive integers such that $f(z)-\alpha = (z-a_j)^{m_j} g_j(z)$, where $g_j(a_j) \neq 0$.

SOLUTION 2. We have

$$f(a_j) = \alpha \qquad \text{for } j = 1, 2, \ldots, n.$$

If $h(z) = f(z) - \alpha$, then $h(a_j) = 0$ for $j = 1, 2, \ldots, n$. Hence we have,

$$h(z) = (z - a_1)^{m_1} g_1(z).$$

Also,

$$h(a_2) = 0 \implies g_1(z) = (z - a_2)^{m_2} g_2(z).$$

$$\vdots$$

$$h(a_n) = 0 \implies g_{n-1} = (z - a_n)^{m_n} g(z).$$

i.e., $h(z) = (z-a_1)^{m_1} \cdots (z-a_n)^{m_n} g(z)$, where g does not vanish in Ω. Then in $\Omega \backslash \{a_1, a_2, \ldots, a_n\}$,

$$\int_\gamma \frac{h'(z)}{h(z)} dz = \int_\gamma \frac{f'(z)}{f(z) - \alpha} dz.$$

Now it is left as an exercise to the reader to verify that

$$\frac{h'(z)}{h(z)} = \frac{m_1}{(z - a_1)} + \frac{m_2}{(z - a_2)} + \cdots + \frac{m_n}{(z - a_n)} + \frac{g'(z)}{g(z)}.$$

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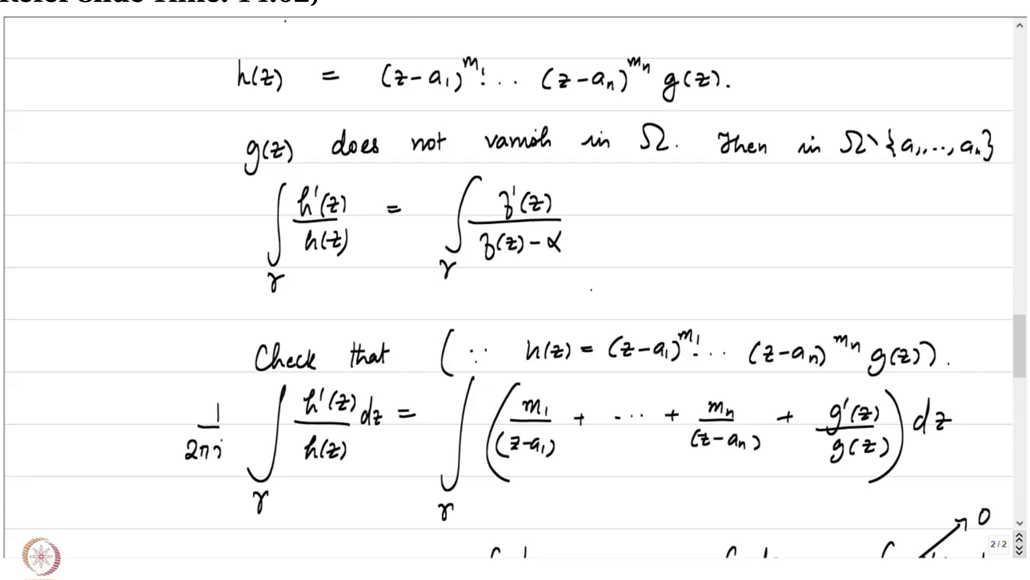

Now,

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}dz = \frac{1}{2\pi i}\int_\gamma \left(\frac{m_1}{(z-a_1)} + \frac{m_2}{(z-a_2)} + \cdots + \frac{m_n}{(z-a_n)} + \frac{g'(z)}{g(z)}\right) dz$$
$$= \frac{1}{2\pi i}\left(m_1\int_\gamma \frac{dz}{(z-a_1)} + m_2\int_\gamma \frac{dz}{(z-a_2)} + \cdots + m_n\int_\gamma \frac{dz}{(z-a_n)} + \int_\gamma \frac{g'(z)}{g(z)}dz\right).$$

By Cauchy's theorem, we have

$$\int_\gamma \frac{g'(z)}{g(z)}dz = 0$$

Also,

$$\frac{m_j}{2\pi i}\int_\gamma \frac{dz}{(z-a_j)} = m_j W_\gamma(a_j) \qquad \text{for } j = 1, 2\ldots, n.$$

Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}dz = \frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}dz = \sum_{j=1}^{n} m_j W_\gamma(a_j).$$

PROBLEM 3. Let Ω be an open connected set containing the origin and $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic such that $f'(0) \neq 0$. Then prove that there exists a neighborhood U of 0 and a function g holomorphic on U such that

$$f(z^n) = f(0) + (g(z))^n \qquad \text{for } n > 0.$$

SOLUTION 3. Let us assume $n > 1$. Put $h(z) = f(z^n) - f(0)$. Then $h'(z) = f'(z^n)\left(nz^{n-1}\right)$ and hence $h'(0) = 0$. Now we are interested in finding the order of zero of h at 0. That is, by Leibniz rule,

$$\frac{d^k}{dz^k}(h'(z))\Big|_{z=0} = \frac{d^k}{dz^k}\left(f'(z^n)\left(nz^{n-1}\right)\right)\Big|_{z=0}$$
$$= \sum_{\ell=0}^{k}\binom{k}{\ell}\frac{d^\ell}{dz^\ell}\left(f'(z^n)\right)\frac{d^{k-\ell}}{dz^{k-\ell}}\left(nz^{n-1}\right)\Big|_{z=0}.$$

Thus, $h^{k+1}(0) = 0$ for $k < n-1$. In other words, for $k < n$, we have $h^k(0) = 0$.

Let us now compute $h^n(0)$. By the computation above,

$$h^n(0) = f'(0)n! \implies h^n(0) \neq 0.$$

Hence $h(z) = z^n \phi(z)$, where $\phi(0) \neq 0$. By a theorem proved earlier, we have

$$h(z) - h(0) = \big(g(z)\big)^n$$

for some function g holomorphic in a neighborhood of 0.

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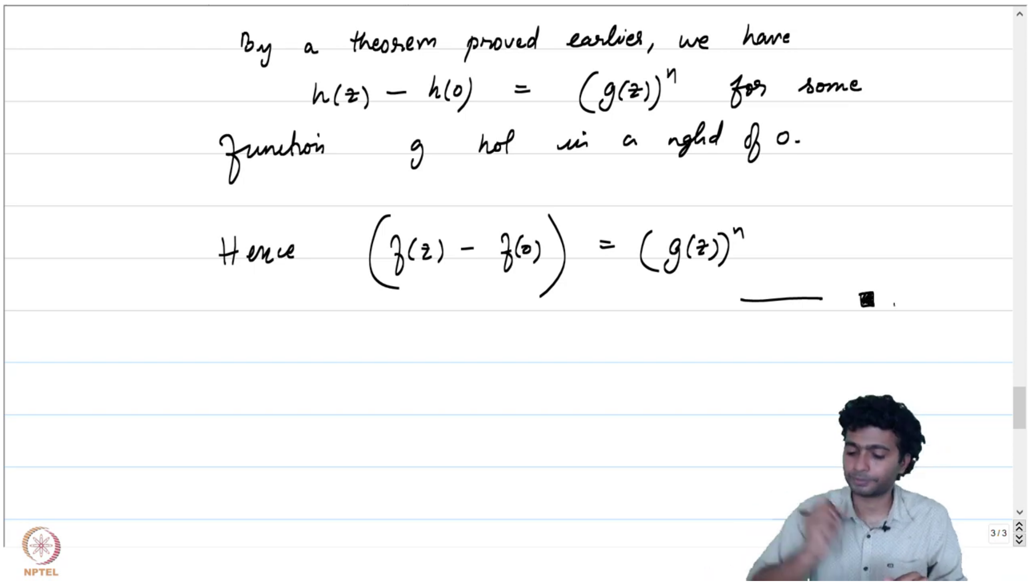

Hence,

$$f(z) - f(0) = \big(g(z)\big)^n.$$

PROBLEM 4. Let $f : \Omega \longrightarrow \mathbb{C}$ be a non-constant holomorphic function on an open connected set Ω. Then $|f(z)|$ does not attain a maximum in Ω.

SOLUTION 4. Let $z_0 \in \Omega$. By open mapping theorem, $f(\Omega)$ is and hence $f(z_0)$ is an interior point. Let $r > 0$ be such that $\overline{D(f(z_0), r)} \subset f(\Omega)$.

Suppose $f(z_0) = Re^{i\theta}$. Then for $0 < s < r$, $f(z_0) + se^{i\theta} \in D(f(z_0), r)$. Now,

$$\left| f(z_0) + se^{i\theta} \right| = \left| Re^{it} + se^{it} \right| = R + s > |f(z_0)|. \tag{1}$$

Since $D(f(z_0), r) \subseteq \Omega$, there exists $w \in \Omega$ such that $f(w) = f(z_0) + se^{i\theta}$. By (1), we have $|f(w)| > |f(z_0)|$. Since the choice of z_0 was arbitrary, for any point $z \in \Omega$, we can get hold of a $w \in \Omega$ with $|f(w)| > |f(z)|$. Hence the maximum cannot be attained in Ω.