Complex Analysis

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Lecture No – 30

Open Mapping Theorem

In this lecture we will explore the local behavior of a non-constant holomorphic function f defined on an open connected set Ω. We will prove that given a point $z_0 \in \Omega$ there exists a neighborhood U of z_0 such that in the punctured neighborhood $U \setminus \{z_0\}$, our function f is an m to 1 mapping for some $m \in \mathbb{N}$.

We will come to that. But, before that, let us prove a special case of the inverse function theorem which states that, if the derivative of the function f does not vanish at a point z_0, then the function f is locally invertible in a neighborhood of z_0 and further we have a holomorphic inverse.

Let us begin this lecture by proving a preparatory lemma which is needed to prove the inverse function theorem.

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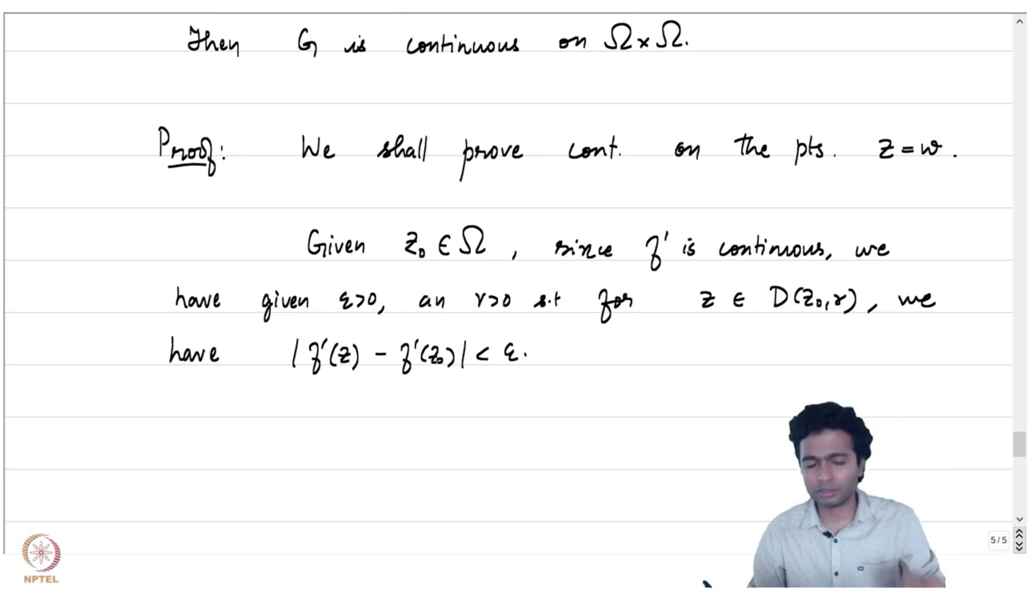

Lemma 1. *Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function on an open set $\Omega \subseteq \mathbb{C}$. Then define $G : \Omega \times \Omega \longrightarrow \mathbb{C}$ given by*

$$G(z,w) := \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{when } z \neq w \\ f'(z) & \text{when } z = w. \end{cases}$$

Then G is continuous on $\Omega \times \Omega$.

Proof. Note that on the points $(z,w) \in \Omega \times \Omega$ where $z \neq w$, continuity of G is trivial.

We shall prove the continuity of G on the points $z = w$. Given $z_0 \in \Omega$, since f' is continuous, if given an $\epsilon > 0$, we have an $r > 0$ such that for $z \in D(z_0, r)$, we have $\left| f(z) - f(z_0) \right| < \epsilon$.

Pick $z, w \in D(z_0, r)$ where $z \neq w$. Let $\gamma(t) = (1-t)z + tw$ for $t \in [0,1]$. Notice that $\gamma'(t) = w - z$. Then,

$$\begin{aligned} \int_0^1 f'\big(\gamma(t)\big)\,dt &= \frac{1}{w-z}\int_0^1 f'\big(\gamma(t)\big)\gamma'(t)\,dt \\ &= \frac{1}{w-z}\int_\gamma f'(z)\,dz \\ &= \frac{f(w)-f(z)}{w-z} \\ &= G(z,w). \end{aligned}$$

Consider

$$\begin{aligned} |G(z,w) - G(z_0,z_0)| &= \left| \int_0^1 f'\big(\gamma(t)\big)\,dt - \int_0^1 f'(z_0)\,dt \right| \\ &= \left| \int_0^1 \big(f'\big(\gamma(t)\big) - f'(z_0)\big)\,dt \right| \\ &\leq \epsilon. \end{aligned}$$

Hence G is continuous at $(z_0, z_0) \in \Omega \times \Omega$. □

Theorem 2. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function. Suppose $z_0 \in \Omega$ be such that $f'(z_0) \neq 0$. Then there exists a neighborhood U of z_0 in Ω such* *that $f \restriction_U$ is injective. Furthermore, $V = f(U)$ is an open set and the inverse $g : V \longrightarrow U$ of f is holomorphic.*

PROOF. Since $f'(z_0) \neq 0$, let $\epsilon = \dfrac{|f'(z_0)|}{2}$. By Lemma 1, there exists a neighborhood U of z_0 such that for $z, w \in U$ and $z \neq w$, we have

$$\left|\frac{f(z)-f(w)}{z-w} - f'(z_0)\right| < \frac{\left|f'(z_0)\right|}{2}$$

$$\frac{\left|f'(z_0)\right|}{2} > \left|f'(z_0)\right| - \left|\frac{f(z)-f(w)}{z-w}\right|$$

(1) $$\implies |f(z)-f(w)| > \frac{\left|f'(z_0)\right|}{2}|z-w|.$$

If $z \neq w$ we have $|f(z)-f(w)| > 0$. That is $f(z) \neq f(w)$. Hence $f \restriction U$ is injective.

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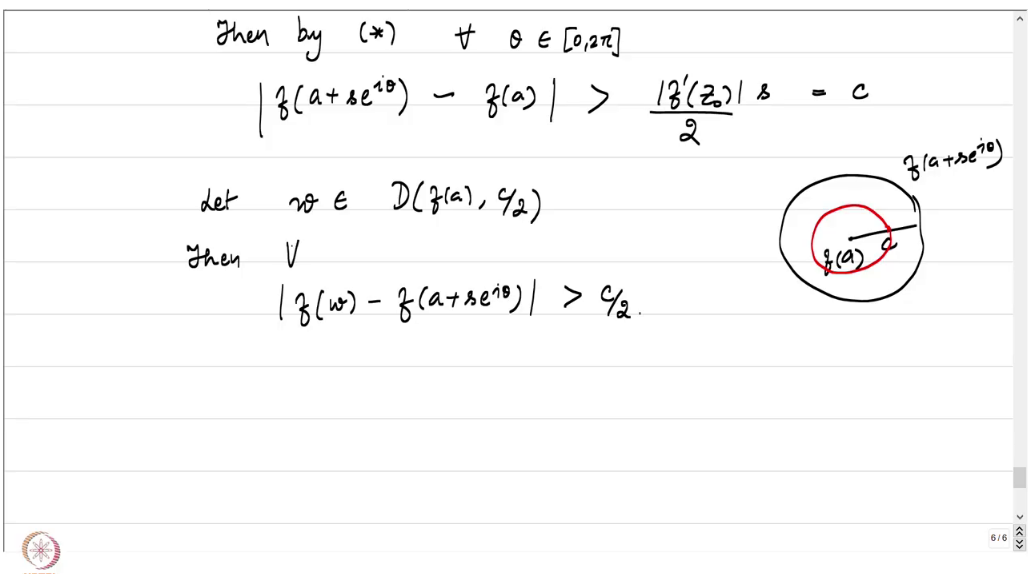

Next we want to show that V, the image of U under f, is an open set. Let $a \in U$. Then there exists $s > 0$ such that $\overline{D(a,s)} \subseteq U$. Then by (1), for every $\theta \in [0, 2\pi]$,

$$\left|f\left(a+se^{i\theta}\right) - f(a)\right| > \frac{|f'(z_0)|s}{2} = c$$

where c is a positive constant. That is $f(a+se^{i\theta})$ is outside the disc of radius c around $f(a)$ for every $\theta \in [0,2\pi]$.

Let $w_0 \in D\left(f(a), \frac{c}{2}\right)$. Then for every $\theta \in [0,2\pi]$, we have,

$$\left| f(w_0) - f\left(a+se^{i\theta}\right) \right| > \frac{c}{2}.$$

Suppose there does not exists $z \in U$ such that $f(z) = w_0$. Then consider $f(z) - w_0$, which is holomorphic and non-vanishing on U. Hence $g(z) = \dfrac{1}{f(z) - w_0}$ is holomorphic on U.

By maximum principle applied to g and compact set $\overline{D(a,s)}$,

$$|g(z)| \leq \sup_{\theta \in [0,2\pi]} \left| g\left(a+se^{i\theta}\right) \right| \qquad \text{for all } z \in D(a,s).$$

Hence,

$$\begin{aligned} \frac{1}{|f(z) - w_0|} &\leq \inf_{\theta \in [0,2\pi]} \frac{1}{\left| f\left(a+se^{i\theta}\right) - w_0 \right|} \\ \implies \inf_{\theta \in [0,2\pi]} \left| f\left(a+se^{i\theta}\right) - w_0 \right| &\leq |f(z) - w_0| \end{aligned}$$

Since for each $\theta \in [0,2\pi]$, $\left| f(w_0) - f\left(a+se^{i\theta}\right) \right| > \dfrac{c}{2}$, we have,

$$\inf_{\theta \in [0,2\pi]} \left| f(w_0) - f\left(a+se^{i\theta}\right) \right| \geq \frac{c}{2}$$

Then,

$$|f(a) - w_0| \geq \frac{c}{2}$$

which is a contradiction to our choice of $w_0 \in D\left(f(a), \frac{c}{2}\right)$. Hence $f(a)$ is an interior point.

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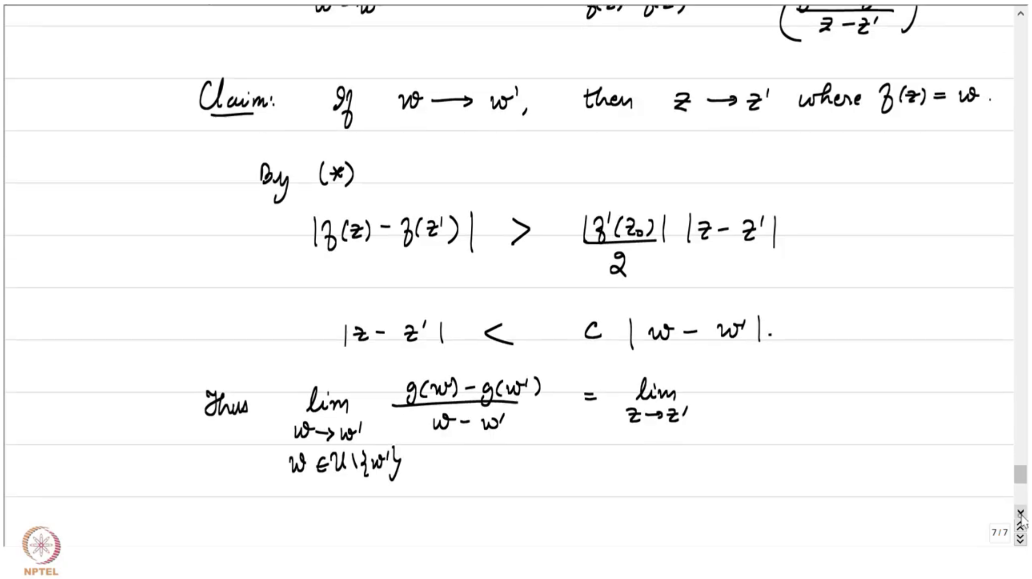

Since $a \in U$ was arbitrary, $V = f(U)$ is an open set. Therefore $f\restriction_U : U \longrightarrow V$ is bijective. Let $g : V \longrightarrow U$ be its inverse. Now it remains to show that this map g is holomorphic on V.

Let $w' \in V$. For $w \in V$ and $w \neq w'$, let $z, z' \in U$ be such that $f(z) = w$, $f(z') = w'$,

$$\frac{g(w)-g(w')}{w-w'} = \frac{z-z'}{f(z)-f(z')} = \frac{1}{\left(\dfrac{f(z)-f(z')}{z-z'}\right)}.$$

Claim: If $w \to w'$, then $z \to z'$ where $f(z) = w$.

By (1),

$$|f(z)-f(z')| > \frac{|f'(z_0)|}{2}|z-z'|$$

$$|z-z'| < C|w-w'|$$

where $C = \dfrac{2}{|f(z_0)|}$. Then the claim is verified, if $w \to w'$, then $z \to z'$.

Thus,

$$\lim_{\substack{w\to w'\\ w\in V\setminus\{w'\}}} \frac{g(w)-g(w')}{w-w'} = \lim_{\substack{z\to z'\\ z\in U\setminus\{z'\}}} \frac{1}{\left(\dfrac{f(z)-f(z')}{z-z'}\right)} = \frac{1}{f'(z')}.$$

Hence g is holomorphic on V. $\square$

EXAMPLE 3. Consider the map $\pi_m : \mathbb{C} \longrightarrow \mathbb{C}$ given by $\pi_m(z) = z^m$ where m is a positive integer. Then, for $z_0 \neq 0$, $\pi'_m(z_0) \neq 0$. Then by Theorem 2, π_m is a local biholomorphic function, i.e, π_m is bijective holomorphic function on a neighborhood of each non-zero point with a holomorphic inverse function.

In particular, If V is an open set which does not contain the origin, then $U = \pi_m(V)$ is also an open set.

Recall that if $w = re^{i\theta} \neq 0$, then $z^m = w$ has roots given by $z_k = r^{1/m} e^{i\left(\frac{\theta}{m} + \frac{2\pi k}{m}\right)}$, $k = 0, 1 \ldots, m-1$. Hence $\pi_m(D(0,r)) = D(0, r^m)$.

THEOREM 4 (Local Behaviour of Holomorphic Functions). *Let $f : \Omega :\longrightarrow \mathbb{C}$ be a non-constant holomorphic function on an open connected set Ω . Let $z_0 \in \Omega$ and $w_0 = f(z_0)$. Then there exists a neighborhood U of z_0 and bijective holomorphic function ϕ on U such that $f(z) = w_0 + \left(\phi(z)\right)^m$ for $z \in U$ and some integer $m > 0$. Moreover ϕ maps U onto $D(0, r)$ for some $r > 0$.*

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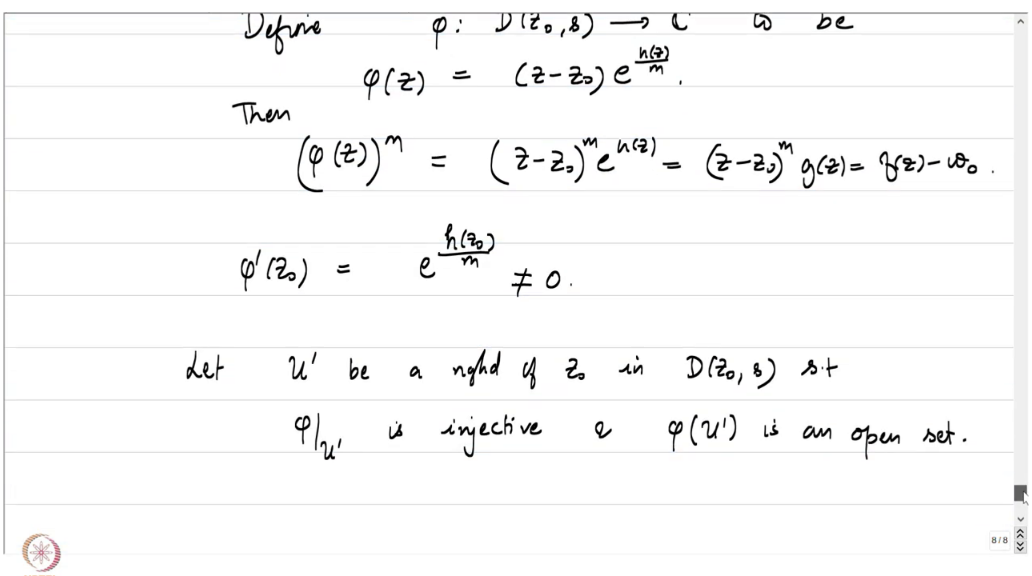

PROOF. Since $f(z_0) = w_0$, we know that $f(z) - w_0$ vanishes at z_0. By using the factorization theorem,

$$f(z) - w_0 = (z - z_0)^m g(z)$$

for some $m > 0$ and such that $g(z_0) \neq 0$.

Let $s > 0$ be such that $g(z) \neq 0$ on $D(z_0, s)$. Since $D(z_0, s)$ is simply connected and g does not vanish on $D(z_0, s)$, there exists a holomorphic function h on $D(z_0, s)$ such that $g(z) = e^{h(z)}$ on $D(z_0, s)$.

Define $\phi : D(z_0, s) \longrightarrow \mathbb{C}$ to be

$$\phi(z) = (z - z_0) e^{\frac{h(z)}{m}}.$$

Then

$$\left(\phi(z)\right)^m = (z - z_0)^m e^{h(z)} = (z - z_0)^m g(z) = f(z) - w_0.$$

Also note that $\phi'(z_0) \neq 0$. Let U' be a neighborhood of z_0 in $D(z_0, s)$ such that $\phi \restriction_{U'}$ is injective and $\phi(U')$ is an open set. Since $\phi(z_0) = 0$, there exists $r > 0$ such that $D(0, r) \subseteq \phi(U')$. Define $U = \phi^{-1}(D(0, r))$.

On U, we have

$$f(z) - w_0 = \left(\phi(z)\right)^m.$$

Also ϕ is bijective and holomorphic on U with $\phi(U) = D(0, r)$. □

Notice that $f(z) = w_0 + \left(\pi_m \circ \phi\right)(z)$, hence f is an m to 1 mapping on $U \setminus \{z_0\}$.

THEOREM 5 (Open Mapping Theorem). *Let $f : \Omega \longrightarrow \mathbb{C}$ be a non-constant holomorphic function on an open connected set Ω. Then $f(\Omega)$ is an open set.*

PROOF. Let $z_0 \in \Omega$ and $w_0 = f(z_0)$. By Theorem 4, there exists a neighborhood U of z_0 such that $f(z) = w_0 + \left(\phi\right)^m$ on U and such that $\phi(U) = D(0, r)$. Hence $\pi_m \circ \phi(U) = D(0, r^m)$. Then $f(U) = D(w_0, r^m) \implies w_0$ is an interior point of $f(\Omega)$. Therefore $f(\Omega)$ is open. □