Complex Analysis

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Lecture – 3

Topology on the Complex Plane

In the last lecture, we defined a metric on the field of complex numbers. The metric was defined using an absolute value, which was a natural extension of the notion of absolute value on the field of real numbers. The absolute value turned out to be the square root of the field norm and we remarked that because of this, it does not matter which copy of the complex field that we work on, the analysis will be consistent.

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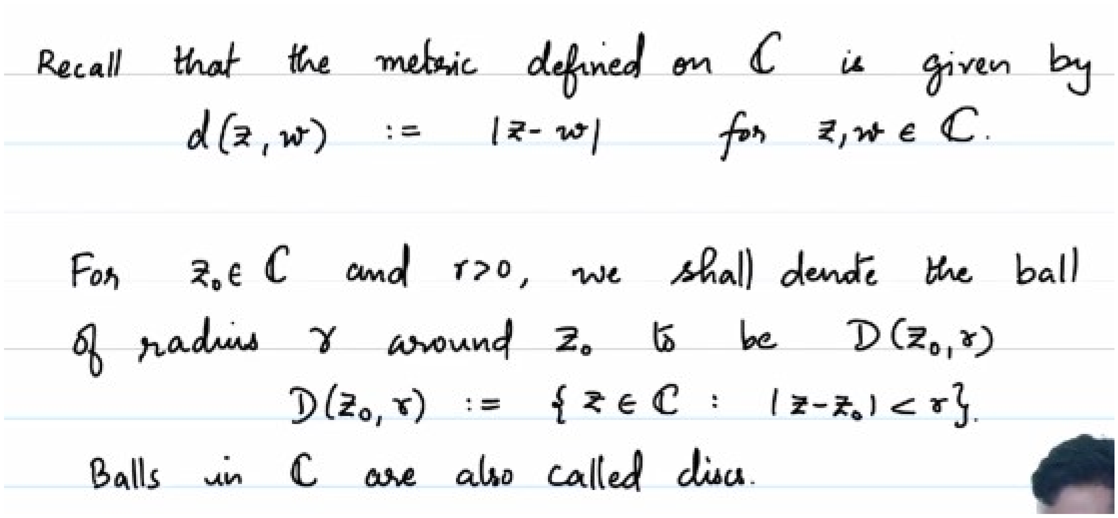

Let us start by recalling the metric that we defined. Recall that the metric defined on $\mathbb{C}$ is given by $d(z, w) := |z - w|, z, w \in \mathbb{C}$. In this lecture, we will do a quick recap of the familiar topological notions on the complex plane. The material will be something which you would have already seen in multivariate real analysis course. The intention is also to set the notations for the rest of this course. First thing to do is describing the balls in $\mathbb{C}$. For a point $z_0 \in \mathbb{C}$ and > 0, denote the ball of radius r around z_0 to be $D(z_0, r)$. Recall that $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$. The D here is actually for disc, since balls in C

are also called discs.

If we are to work with general metric space, which we will do sometimes in this lecture itself, we will use the word B instead of D, which should mean the same thing.

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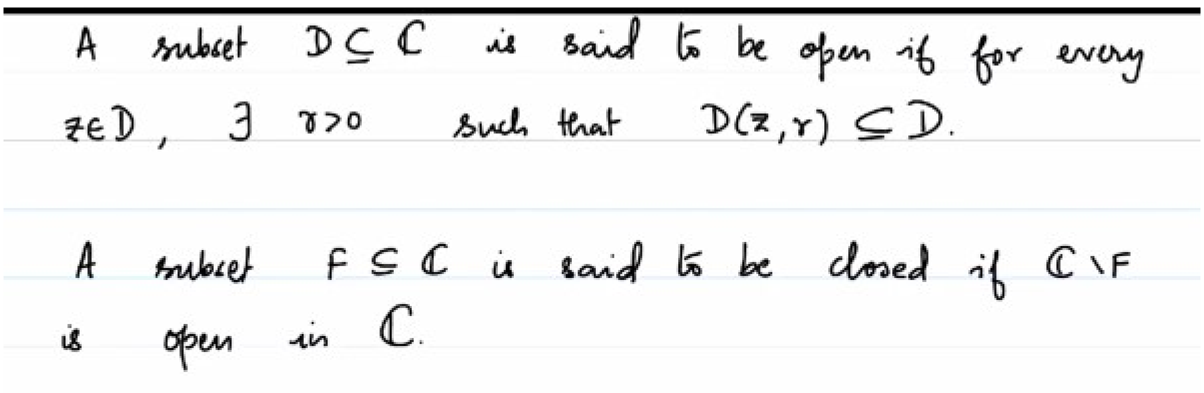

A subset $D \subseteq \mathbb{C}$ is said to be **open** if for every $z \in D$, $\exists r > 0$ such that $D(z, r) \subseteq D$. This can be restated in another way as, if every point in D is an interior point of D, we say that D is open.

A subset $F \subseteq \mathbb{C}$ is said to be **closed** if $\mathbb{C}$, the complement of F, $\mathbb{C} \setminus F$ is open in $\mathbb{C}$.

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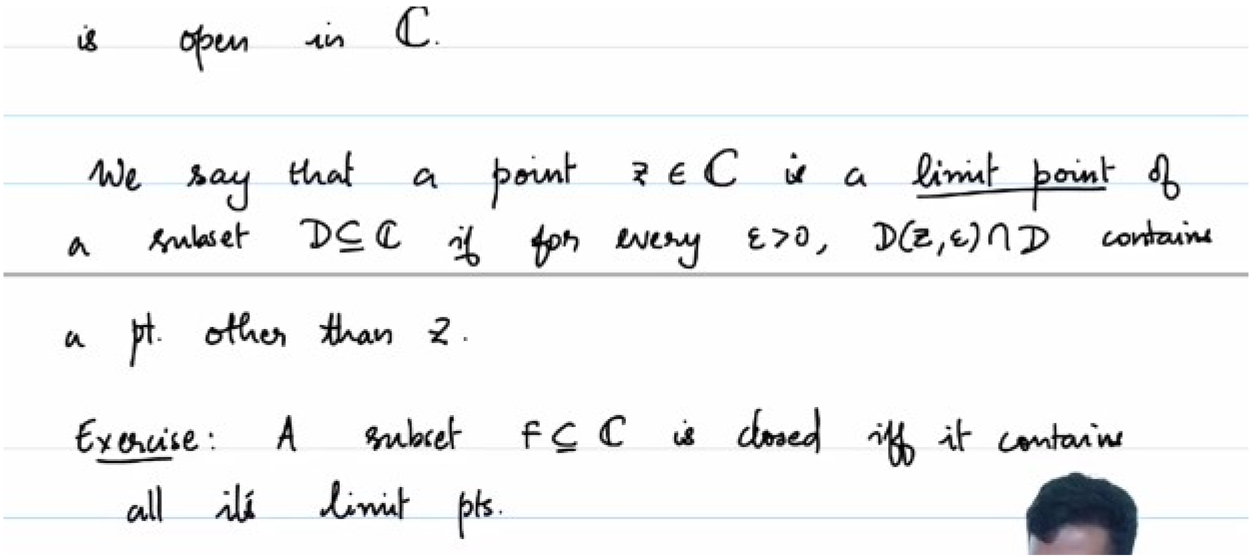

A point $z \in \mathbb{C}$ is said to be a **limit point** of a subset $D \subseteq \mathbb{C}$, if for every $\epsilon > 0$, $D(z,\epsilon) \cap D$ contains a point other than z.

In a metric space, this is also the same as demanding that you have a sequence in D converging to z, where the sequence consists of points other than z.

EXERCISE 1. A subset $F \subseteq \mathbb{C}$ is closed if and only if it contains all its limit points.

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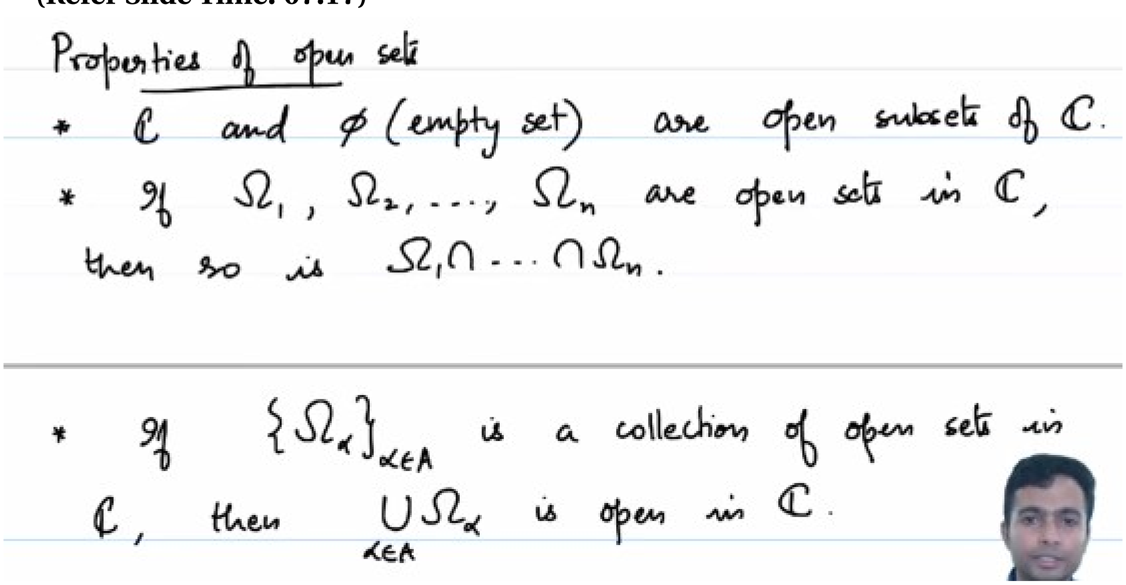

Properties of open sets

(1) $\mathbb{C}$ and $\varnothing$(empty set) are open subsets of $\mathbb{C}$.

(2) If $\Omega_1, \Omega_2, \ldots \Omega_n$ are open sets in $\mathbb{C}$, then $\Omega_1 \cap \Omega_2 \cap \ldots \Omega_n$ are also open. That is finite intersection of open sets is open.

(3) If $\{\Omega_\alpha\}_{\alpha \in A}$ is a collection of open sets of $\mathbb{C}$, then $\bigcup\limits_{\alpha \in A} \Omega_\alpha$ is open in $\mathbb{C}$.

Verification of these properties left as an exercise to the reader. Check that each of the sets mentioned satisfy the definition of open set.

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The collection of all open sets on $\mathbb{C}$ form a topology on $\mathbb{C}$.

Hence we can say that the collection of all open sets on $\mathbb{C}$ form a topology on $\mathbb{C}$.

Let $D \subseteq \mathbb{C}$, we define the **interior** of D, denoted by D^o, to be the union of all open sets in $\mathbb{C}$ which are contained in D.

That is, interior of D, $D^o := \bigcup\{\Omega : \Omega \subseteq D \text{ and } \Omega \text{ is open in } \mathbb{C}\}$.

From the properties stated above, it is clear that D^o is an open set.

We define **closure** of D to be the set, $\overline{D} := \bigcap\{F : F \subseteq D \text{ and } F \text{ is closed in } \mathbb{C}$.

Using De Morgan's law, we can say that the arbitrary intersection of closed sets will again be a closed set. Hence $\overline{D}$ is a closed set. The interior is the biggest open set contained in D and closure, $\overline{D}$, is the smallest closed set which contains D.

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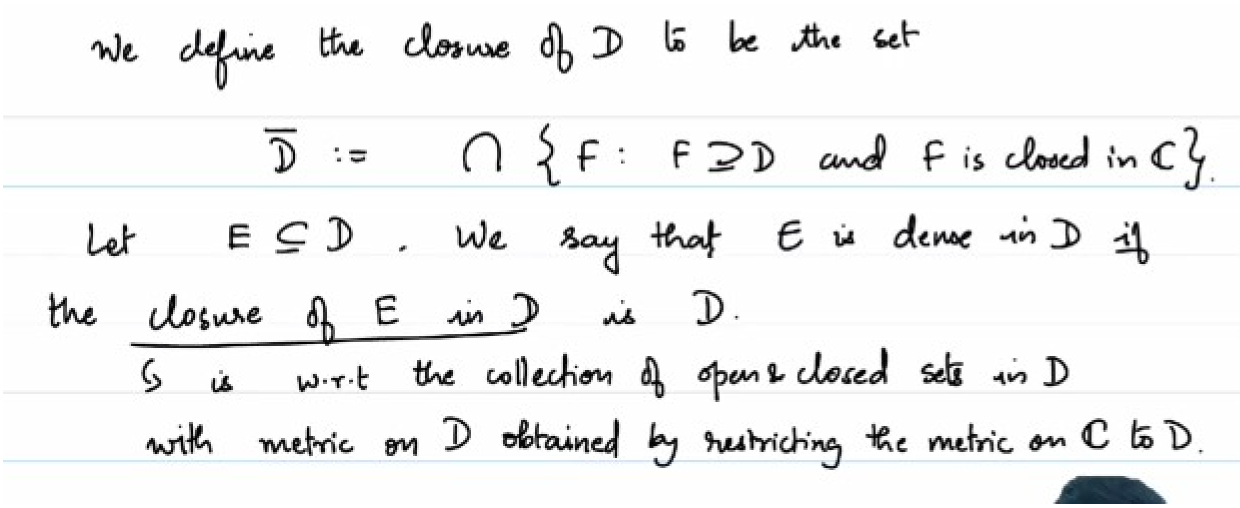

We define the closure of D to be the set

$$\overline{D} := \bigcap \{F : F \supseteq D \text{ and } F \text{ is closed in } \mathbb{C}\}.$$

Let $E \subseteq D$. We say that E is dense in D if the closure of E in D is D.

↳ is w.r.t the collection of open & closed sets in D with metric on D obtained by restricting the metric on $\mathbb{C}$ to D.

Let $E \subseteq D$. We say that E is **dense** in D, if the closure of E in D is D, where the closure of E in D, in general metric space, is with respect to the collection of open and closed sets in D with metric on D obtained by restricting the metric on $\mathbb{C}$ to D.
Anyway in the case of our complex plane, closure of E in D is the intersection of closure of E in $\mathbb{C}$ and D.

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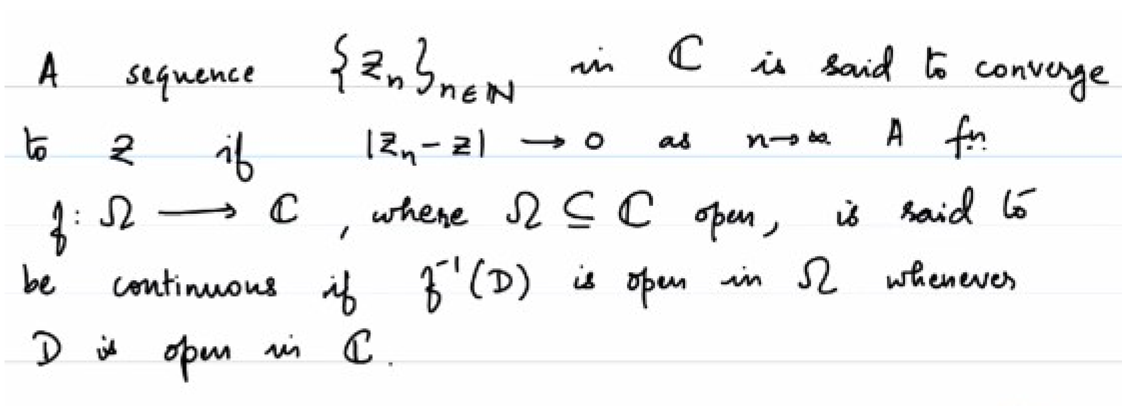

A sequence $\{z_n\}_{n \in \mathbb{N}}$ in $\mathbb{C}$ is said to converge to z if $|z_n - z| \to 0$ as $n \to \infty$. A fn. $f : \Omega \longrightarrow \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$ open, is said to be continuous if $f^{-1}(D)$ is open in Ω whenever D is open in $\mathbb{C}$.

A sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$ is said to **converge** to z if $|z_n - z| \longrightarrow 0$ as $n \longrightarrow \infty$.
Let $\Omega \subseteq \mathbb{C}$ be open and a function, $f : \Omega \longrightarrow \mathbb{C}$ is said to be **continuous** if $f^{-1}(D)$ is open in Ω whenever D is open in $\mathbb{C}$. It is the same as saying that if there is a sequence $\{z_n\}_{n\in\mathbb{N}}$

in Ω which converges to z in Ω, then $f(z_n) \longrightarrow f(z)$.

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be continuous if $f^{-1}(D)$ is open in Ω whenever D is open in $\mathbb{C}$.

$+ : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ and $\times : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$.

$$\lim_{n\to\infty} (z_n + w_n) = \lim_{n\to\infty} z_n + \lim_{n\to\infty} w_n$$

$$\lim_{n\to\infty} (z_n . w_n) = \lim_{n\to\infty} z_n \lim_{n\to\infty} w_n.$$

We know that if $\{z_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}}$ are two convergent complex sequences, then $\lim_{n\to\infty}(z_n + w_n) = \lim_{n\to\infty} z_n + \lim_{n\to\infty} w_n$ and $\lim_{n\to\infty}(z_n \times w_n) = \lim_{n\to\infty} z_n \times \lim_{n\to\infty} w_n$. From here we can conclude that the addition and multiplication operation on complex plane is continuous. This can be checked by considering the triangle inequality and the fact that $|zw| = |z||w|$.

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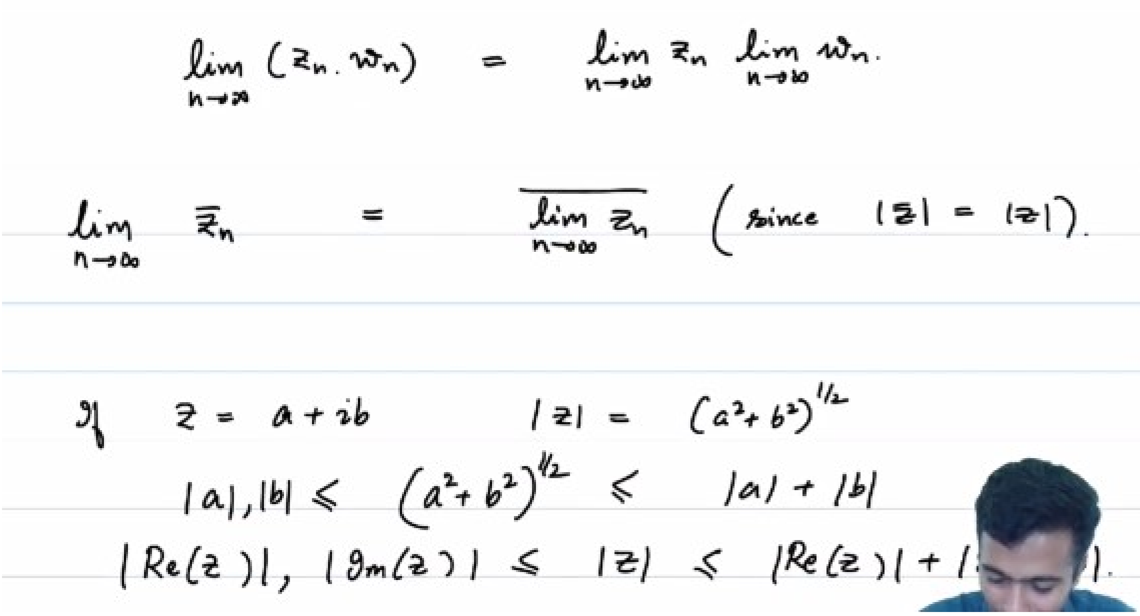

Also notice that $\lim\limits_{n\to\infty} \overline{z_n} = \overline{\lim\limits_{n\to\infty} z_n}$ which tells us that conjugation is also a continuous function, which follows form the observation that $|\overline{z}| = |z|$.

Our next aim is to show that the field of complex number is complete. For that we use the fact that $\mathbb{R}$ is complete.

If $z = a + ib$, then the absolute value of z, $|z| = \sqrt{a^2 + b^2}$. Then
$|a|, |b| \leq \sqrt{a^2 + b^2} \leq |a| + |b| \implies |\mathfrak{Re}(z)|, |\mathfrak{Im}(z)| \leq |z| \leq |\mathfrak{Re}(z)| + |\mathfrak{Im}(z)|$.

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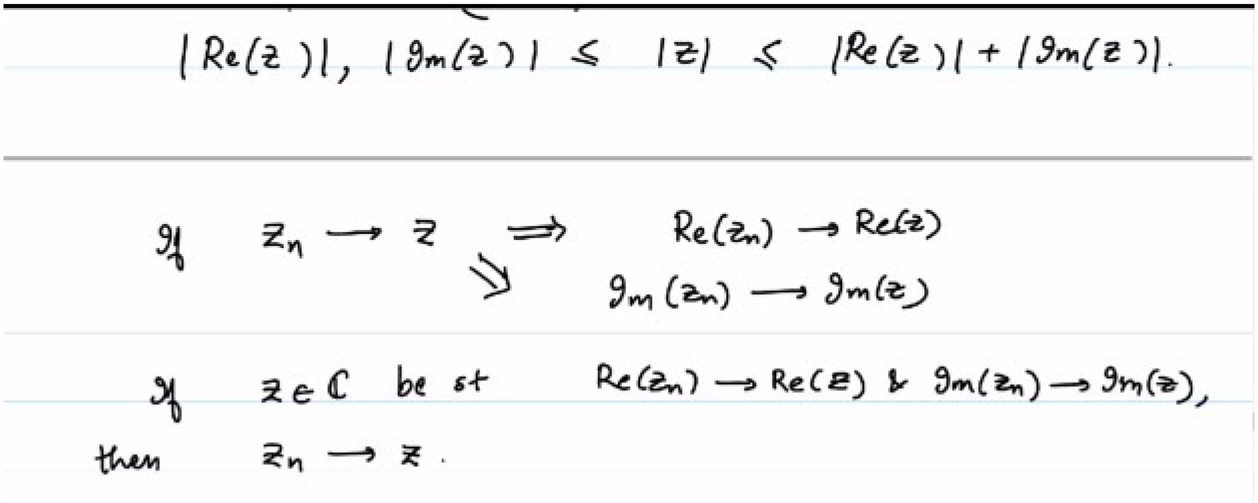

Because of these inequalities, if $\{z_n\}$ is a sequence of complex numbers which converges to z, then $\mathfrak{Re}(z_n) \to \mathfrak{Re}(z)$, similarly $\mathfrak{Im}(z_n) \to \mathfrak{Im}(z_n)$.
We can say it another way also, that if $\{z\}$ is sequence of complex numbers converges to a point $z \in \mathbb{C}$ if and only if $\mathfrak{Re}(z_n) \to \mathfrak{Re}(z)$ and $\mathfrak{Im}(z_n) \to \mathfrak{Im}(z)$.

COROLLARY 2. *The field of complex number $\mathbb{C}$ is complete with respect to the metric defined as above.*

If $\{z_n\}$ is a Cauchy sequence, then $\{\mathfrak{Re}(z_n)\}$ and $\{\mathfrak{Im}(z_n)\}$ both are Cauchy and by the completeness of $\mathbb{R}$, there exist a, b such that $\mathfrak{Re}(z_n) \to a$ and $\mathfrak{Im}(z_n) \to b$. Let $z = a + ib$. Then $z_n \to z$. Hence every Cauchy sequence has a limit, therefore $\mathbb{C}$ is complete.
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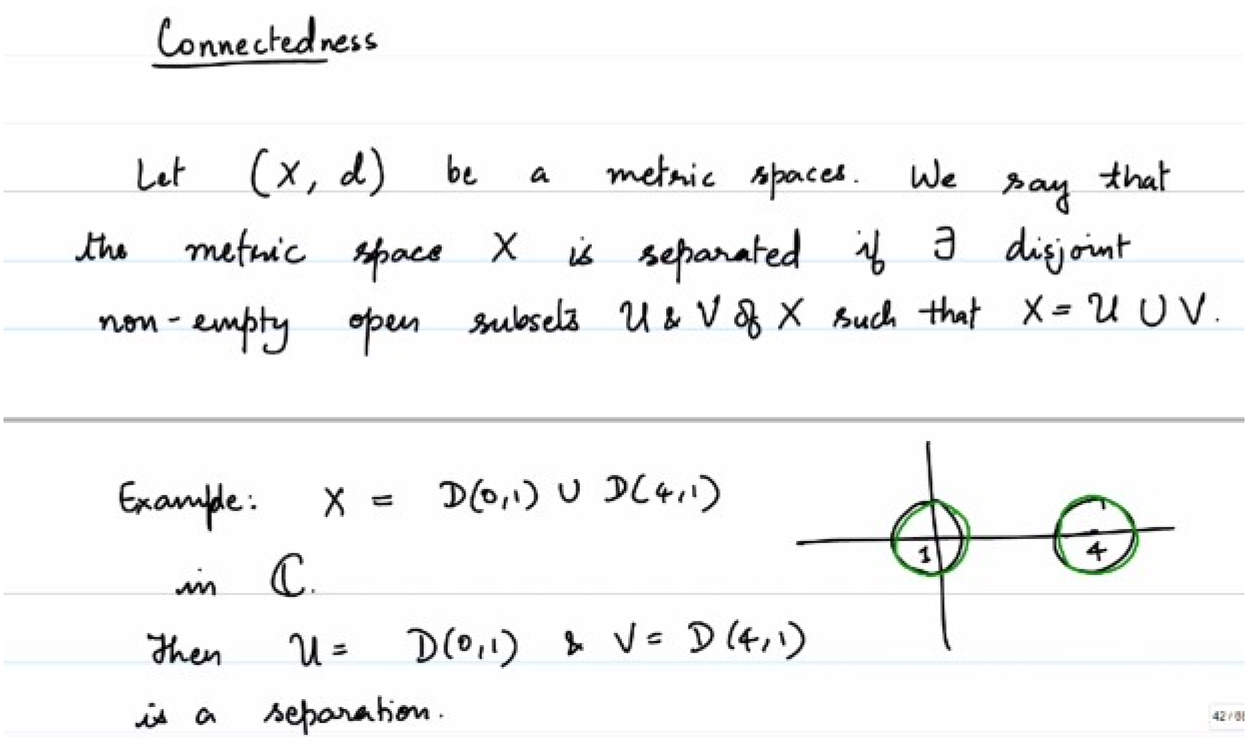

Next important notion is connectedness. We may work with the general metric space rather than $\mathbb{C}$ for defining the connectedness.

Let (X, d) be metric space. We say that X is **separated** if there exist disjoint non-empty open subsets U, V of X such that $X = U \bigcup V$.

For example, let $X \subseteq \mathbb{C}, X = D(0,1) \cup D(4,1)$ and let us use the metric on $\mathbb{C}$ to restrict it to X , then $U = D(0,1)$ and $V = D(4,1)$ is a separation.

Also, if we had taken closed discs instead of open discs, that is if $X' = \overline{D(0,1)} \bigcup \overline{D(4,1)}$ and look at a new metric space. Then again, the reader should verify that $U = \overline{D(0,1)}, V = \overline{D(4,1)}$, that are not open in $\mathbb{C}$, are open in X' in the subspace topology, then $U \cap V = \emptyset$ and the union will give us X'.Hence X' is also separated.

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We say that a metric space is connected if there does not exist a separation of X.

X is connected iff X does not contain a proper subset of X which is both open & closed in X.

Examples: A subset $X \subseteq \mathbb{R}$ is connected iff X is an interval.

We say that a metric space (X,d) is **connected** if there does not exist a separation of X.

We can reformulate the definition as, X is connected if and only if X does not contain a proper subset which is both open and closed and simultaneously in X.

The examples we have mentioned above for the spaces that are separated, $X = D(0,1) \cup D(4,1)$, we can see that $D(0,1)$ is open and closed in X, since $D(0,1)$ is open in $\mathbb{C}$, under subspace topology it is open in X and also the compliment of $D(0,1)$ in X is an open set in X, thus $D(0,1)$ is closed. Similarly $D(4,1)$ is both open and closed in X which implies that X is separated. Similarly in the case of X', $\overline{D(0,1)}$ and $\overline{D(4,1)}$ are both open and closed in X'.

Example: $X \subseteq \mathbb{R}$ is connected if and only if X is an interval.

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Examples: A subset $X \subseteq \mathbb{R}$ is connected iff X is an interval.

Proposition: Let $f : X \to Y$ be a continuous fn between metric spaces. Then $f(x)$ is connected if X is connected.

So, here we have given a characterization of how the connected sets in $\mathbb{R}$ behave. We will not be able to give this kind of a characterization on the complex plane. However, we will give an alternate description of how connected open sets in $\mathbb{C}$ will behave.

PROPOSITION 3. *Let X, Y are metric spaces and $f : X \longrightarrow Y$ is a continuous function. Then $f(x)$ is connected if X is connected.*

PROOF. The proof is by contradiction. Let $Z = f(X) \subseteq Y$. Suppose Z is not connected.Then $Z = A \bigcup B$ where A, B are non-empty, open in Z and $A \cap B = \varnothing$.
Claim: $f^{-1}(A)$ and $f^{-1}(B)$ are open in X.
Since $A = U \cap Z$, U open in $Y, f^{-1}(A) = f^{-1}(U)$, and f is continuous $\implies f^{-1}(U) = f^{-1}(A)$ is open in X. Similarly $f^{-1}(B)$ is open in X. f is surjective onto Z and A, B are non-empty $\implies f^{-1}(A), f^{-1}(B)$ are non-empty. Also $Z = A \cup B \implies f^{-1}(A) \cup f^{-1}(B) = X$ also $f^{-1}(A) \cap f^{-1}(B) = \varnothing$. Then X is separated by $f^{-1}(A)$ and $f^{-1}(B)$ which is a contradiction to our assumption that X is connected. Hence $f(X)$ is connected. □

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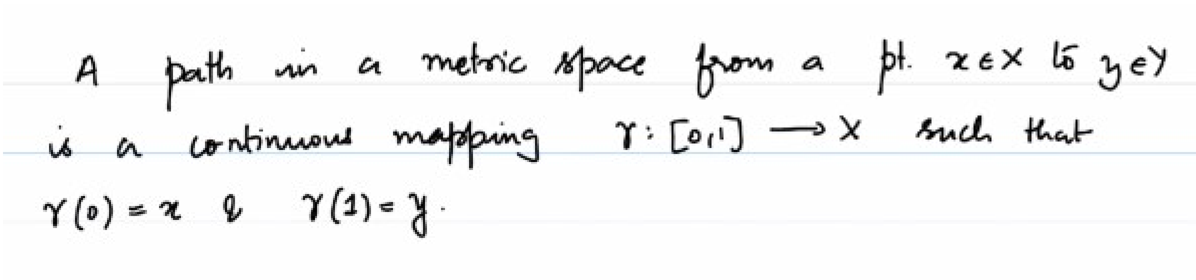

A **path** in a metric space from a point $x \in X$ to $y \in X$ is a continuous map $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

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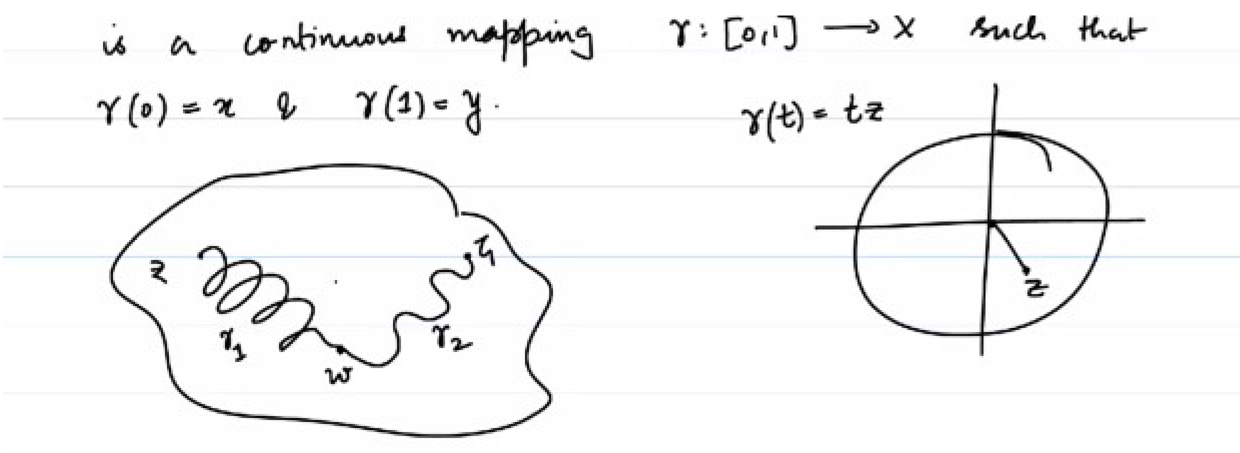

Example: Consider $z \in D(0,1)$ and $\gamma : [0,1] \longrightarrow \mathbb{C}$ such that $\gamma(t) = tz$. Then γ is a straight line from 0 to z.

Let γ_1 be a path from x to y and γ_2 be a path from y to z in a metric space X. Define

$$\sigma(s) = \begin{cases} \gamma_1(2s) \text{ if, } s \in [0, \frac{1}{2}] \\ \gamma_2(2s-1) \text{ if, } s \in [\frac{1}{2}, 1] \end{cases}$$

Then σ is a continuous map from $[0,1]$ to X such that $\sigma(0) = x, \sigma(1) = z$.

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Theorem: An open subset $\Omega \subsetneq \mathbb{C}$ is connected iff for $z, w \in \Omega$, there exists a path from z to w.

Proof: $(\Leftarrow)$ Suppose Ω is separated. Then $\Omega = \mathcal{U} \cup V$. Let $z \in \mathcal{U}$ and $w \in V$.

THEOREM 4. *A open subset $\Omega \subseteq \mathbb{C}$ is connected if and only if there exists a path for every pair of points $z, w \in \Omega$ there exists a path from z to w.*

PROOF. $(\Leftarrow)$ Suppose Ω is separated. Then $\exists U, V$ such that $\Omega = U \cup V$ where U, V are non-empty open subsets of Ω. Let $z \in U$ and $w \in V$. If there exists a path γ form z to w, then $\gamma : [0,1] \longrightarrow X$ is continuous and $\gamma(0) = z$ and $\gamma(1) = w$. Then $[0,1] = f^{-1}(U) \bigcup f^{-1}(V)$, which is a contradiction since $[0,1]$ is connected. Therefore, there does not exists a separation of Ω, hence Ω is connected.
$(\Rightarrow)$ Let us assume that Ω is connected. Fix $z_0 \in \Omega$. Let $A := \{z \in \Omega : \exists$ a path from z to $z_0\}$.
Claim: A is open.
Let $z \in A$. Since Ω is open $\exists\, r > 0$ such that $D(z, r) \subseteq \Omega$. Since $z \in A$, $\exists$ a path γ from z_0 to z. For any $w \in D(z, r)$ consider the path $\gamma_1 : [0,1] \longrightarrow \Omega$ given by $\gamma_1(s) = (1-s)z + sw$. Then γ_1 is a straight line path from z to w and is contained in $D(z, r)$. Define

$$\sigma(s) = \begin{cases} \gamma(2s) \text{ if, } s \in [0, \frac{1}{2}] \\ \gamma_1(2s-1) \text{ if, } s \in [\frac{1}{2}, 1] \end{cases}$$

Hence σ is a path from z_0 to $w \implies w \in A \implies D(z, r) \subseteq A \implies A$ is open.
Claim: A is closed.
Let $z \in \Omega \setminus A$. Since Ω is open, we have $r > 0$ such that $D(z, r) \subseteq \Omega$. If $D(z, r) \cap A \neq \varnothing$,

then $\exists w \in D(z,r) \cap A$. This would give a path from z_0 to z through w contradiction to our assumption that $z \in \Omega \setminus A$. Hence $D(z,r) \cap A = \varnothing \implies \Omega \setminus A$ is open $\implies A$ is closed. Hence A is both open and closed. Therefore either $A = \Omega$ or $A = \varnothing$. Since $z_0 \in A, A \neq \varnothing \implies A = \Omega$. Thus any pair of points in a can be connected by a path. $\square$