Complex Analysis

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Lecture No – 29

Winding Number

In the version of the Cauchy integral formula that we have proved, we have imposed a strict condition on the curve in the question. We demanded that in $\mathbb{C} \setminus \{z_0\}$, the curve is homotopic as closed curve to a circle centered at z_0 with radius r, for some $r > 0$. This is a bit stringent condition to impose on the curve and we would like to circumvent this condition. The route to getting a more general Cauchy integral formula is by introducing the notion of the winding number of a curve.

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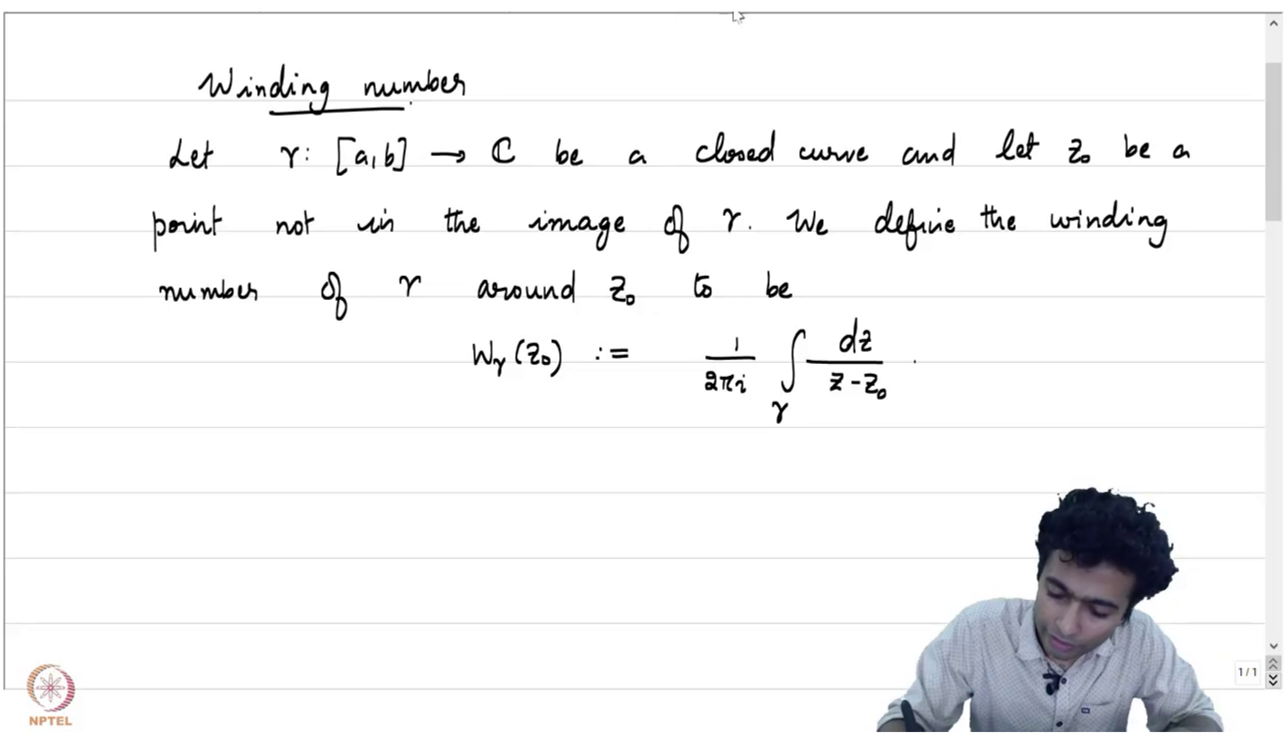

DEFINITION 1 (**Winding Number**). Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a closed curve and let $z_0 \in \mathbb{C} \setminus \gamma([a,b])$. We define the winding number of γ around z_0 to be

$$W_\gamma(z_0) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}.$$

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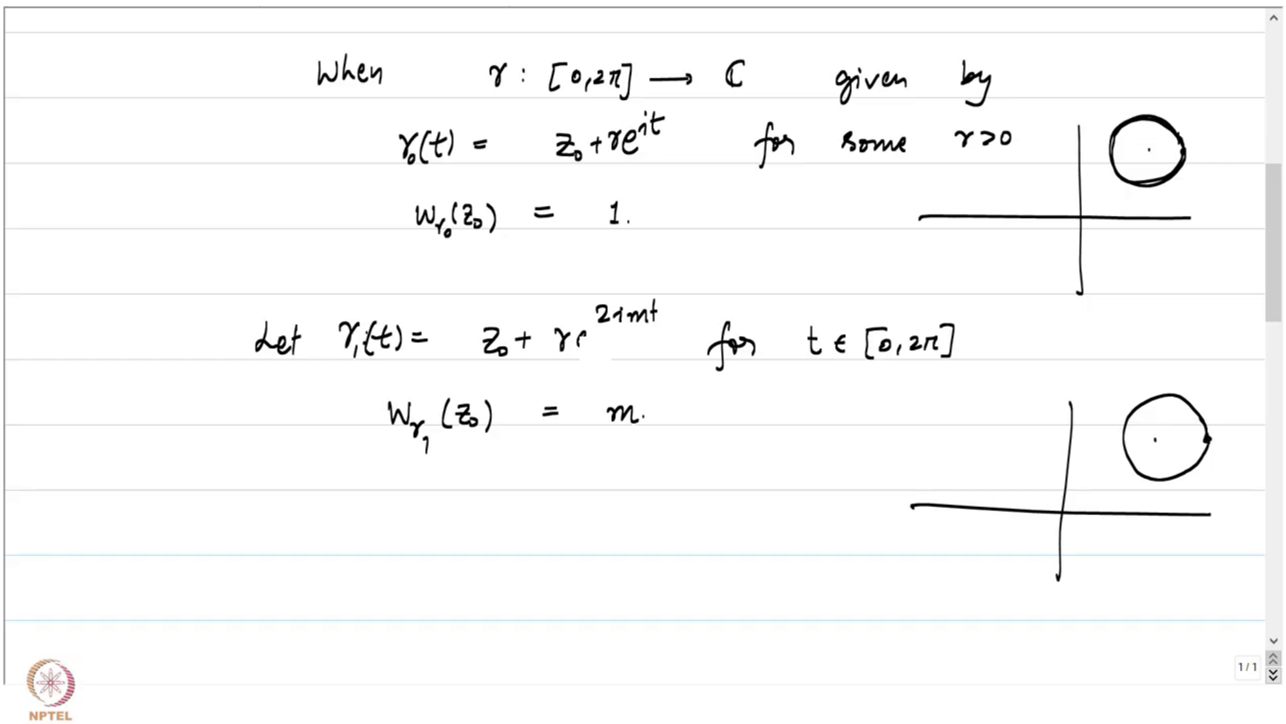

EXAMPLE 1. Let $\gamma_0 : [0,2\pi] \longrightarrow \mathbb{C}$ be given by $\gamma_0(t) = z_0 + re^{it}$ for some $r > 0$. Then by Cauchy integral formula, we have

$$W_{\gamma_0}(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = 1.$$

More generally, let $\gamma_1 : [0,2\pi] \longrightarrow \mathbb{C}$ be given by $\gamma_1(t) = z_0 + re^{imt}$ for some $r > 0$ and $m \in \mathbb{N}$. Then we have $W_{\gamma_1}(z_0) = m$.

Let γ and σ be two closed curves in $\mathbb{C} \setminus \{z_0\}$ such that γ is homotopic as closed curves to a reparametrization of σ in $\mathbb{C} \setminus \{z_0\}$. Then by Cauchy's theorem, we have

$$\int_\gamma \frac{dz}{z - z_0} = \int_\sigma \frac{dz}{z - z_0}.$$

Since $\dfrac{1}{z - z_0}$ is a holomorphic function on $\mathbb{C} \setminus \{z_0\}$,

$$W_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i}\int_\sigma \frac{dz}{z - z_0} = W_\sigma(z_0).$$

PROPOSITION 2. *Let $\gamma_0 : [a,b] \longrightarrow \mathbb{C}$ be a closed curve and z_0 be a point not in γ_0. Suppose $\gamma_1 : [a,b] \longrightarrow \mathbb{C}$ be a closed curve such that*

$$\left|\gamma_0(t) - \gamma_1(t)\right| < \left|\gamma_0(t) - z_0\right|.$$

Then $W_{\gamma_0}(z_0) = W_{\gamma_1}(z_0)$.

PROOF. Define $H : [0,1] \times [a,b] \longrightarrow \mathbb{C}$ by

$$H(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t).$$

Then H is continuous function into $\mathbb{C}$ and notice that at each point $s \in [0,1]$, $\gamma_s : [a,b] \longrightarrow \mathbb{C}$ given by $\gamma_s(t) = H(s,t)$ is a closed curve. Hence H is a homotopy from γ_0 to γ_1.

Claim: $z_0 \notin H([0,1] \times [a,b])$.

$$\begin{aligned}
|H(s,t) - z_0| &= \left|(1-s)\gamma_0(t) + s\gamma_1(t) - z_0\right| \\
&= \left|s\big(\gamma_1(t) - \gamma_0(t)\big) + \gamma_0(t) - z_0\right| \\
&\geq \left|\gamma_0(t) - z_0\right| - s\left|\gamma_1(t) - \gamma_0(t)\right| \\
&\geq \left|\gamma_0(t) - z_0\right| - \left|\gamma_1(t) - \gamma_0(t)\right| \\
&> 0.
\end{aligned}$$

Hence $H(s,t) \neq z_0$ for any $(s,t) \in [0,1] \times [a,b]$. Therefore H is a homotopy of closed curves in $\mathbb{C} \setminus \{z_0\}$ from γ_0 to γ_1 $\implies W_{\gamma_0}(z_0) = W_{\gamma_1}(z_0)$. $\square$

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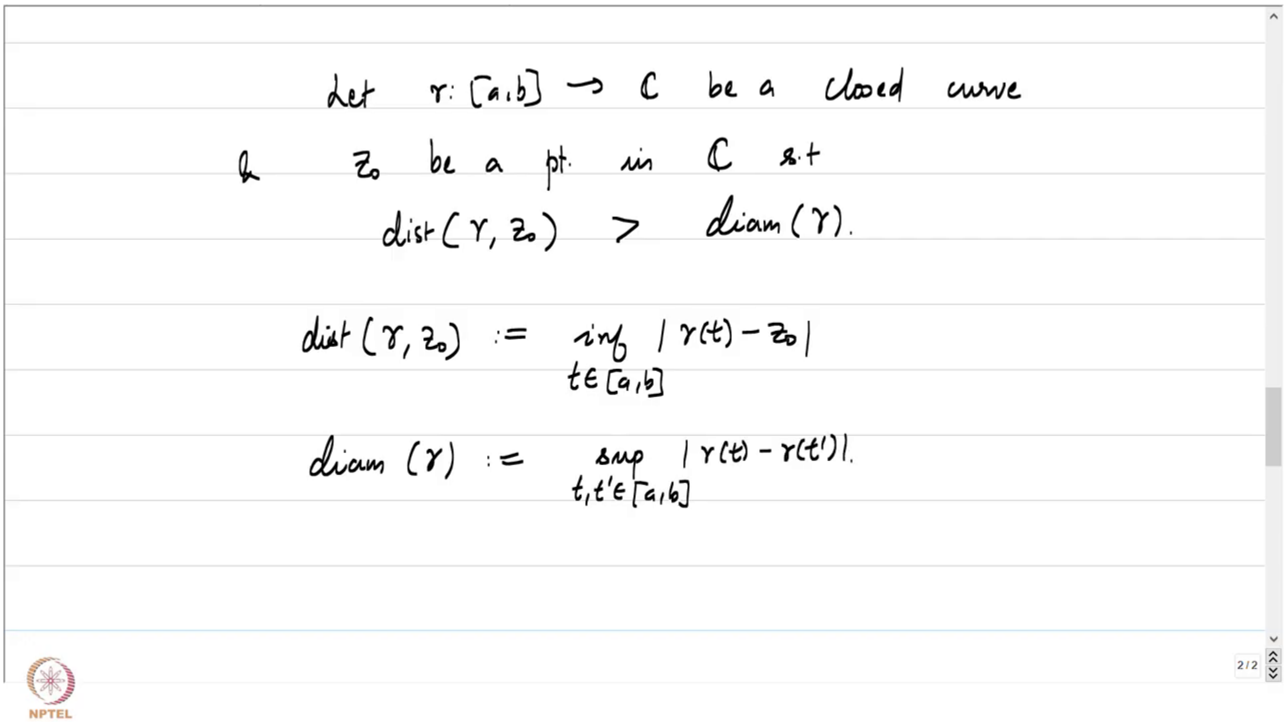

Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a closed curve and z_0 be a point in $\mathbb{C}$ such that

$$\mathrm{dist}(\gamma, z_0) > \mathrm{diam}(\gamma),$$

where $\mathrm{dist}(\gamma, z_0) := \inf_{t \in [a,b]} \left|\gamma(t) - z_0\right|$ and $\mathrm{diam}(\gamma) := \sup_{t,t' \in [a,b]} \left|\gamma(t) - \gamma(t')\right|$.

Let us see what happens when we have such a scenario. Define γ_1 to a constant curve as $\gamma_1(t) = z = \gamma(a)$ for every $t \in [a,b]$. Then,

$$\left|\gamma(t) - \gamma_1(t)\right| < \mathrm{diam}(\gamma) < \mathrm{dist}(\gamma, z_0) \leq \left|\gamma(t) - z_0\right|.$$

Hence by Proposition 2, we have $W_\gamma(z_0) = W_{\gamma_1}(z_0)$. But since γ_1 is a constant curve, $W_{\gamma_1}(z_0) = 0 = W_\gamma(z_0)$. Geometrically it tells that, if z_0 is a point on the complex plane which is far away from the curve γ, then the curve does not wind around z_0, as to be expected.

PROPOSITION 3. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a closed curve and z_0 be a point not in γ. Then there exists $r > 0$ such that for $z \in D(z_0, r)$, $W_\gamma(z) = W_\gamma(z_0)$.*

PROOF. Since $\gamma([a,b])$ is a closed subset of $\mathbb{C}$ and $z_0 \notin \gamma([a,b])$, there exists $r > 0$ such that $D(z_0, r) \cap \gamma([a,b]) = \varnothing$. Let $h \in \mathbb{C}$ be such that $|h| < r$. Define $\gamma_h(t) = \gamma(t) - h$ for $t \in [a,b]$. Then,

$$\left|\gamma_h(t) - \gamma(t)\right| = |h| < r < \left|\gamma(t) - z_0\right|.$$

Hence

$$W_{\gamma_h}(z_0) = W_\gamma(z_0). \tag{1}$$

Now it is left as an exercise to the reader to verify the following equality,

$$W_{\gamma_h}(z_0) = \frac{1}{2\pi i}\int_{\gamma_h} \frac{dz}{z - z_0} = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z - (z_0 + h)} = W_\gamma(z_0 + h).$$

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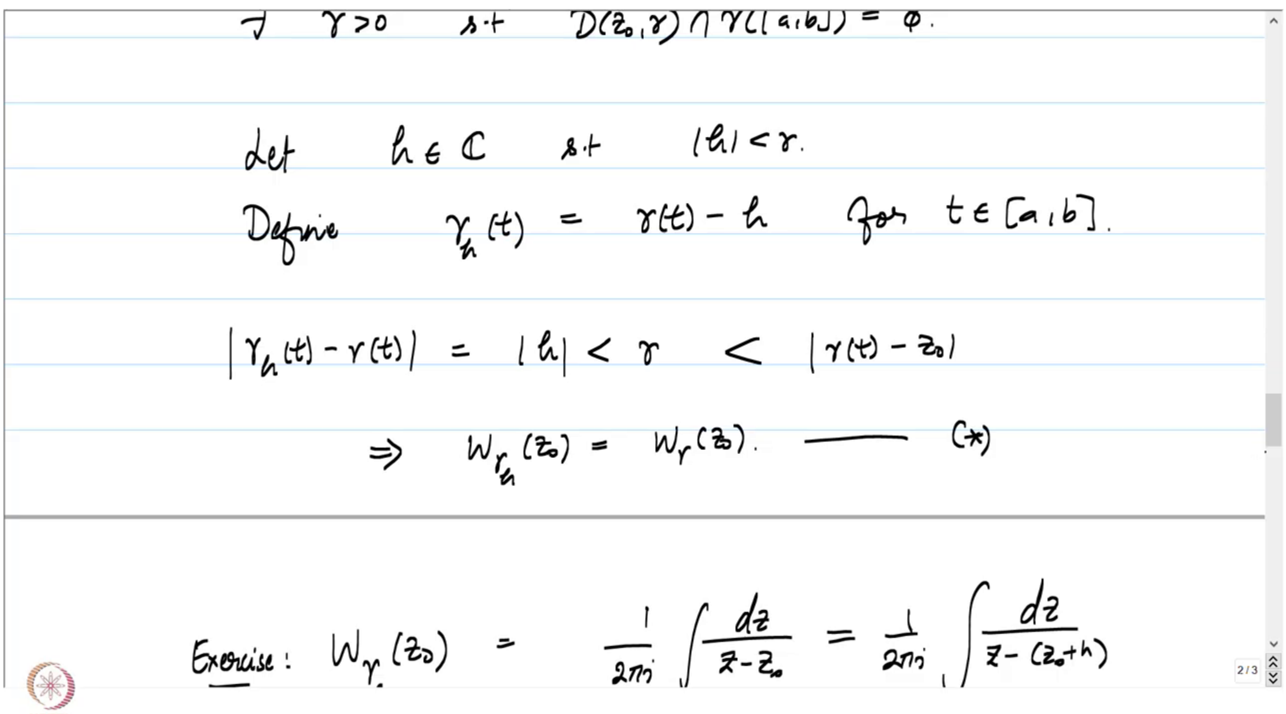

From (1), for each h with $|h| < r$, we have

$$W_\gamma(z_0) = W_\gamma(z_0 + h).$$

Hence W_γ is constant on $D(z_0, r)$. $\square$

THEOREM 4. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a closed curve and z_0 be a point not in the image of γ. Then $W_\gamma(z_0)$ is an integer.*

PROOF. Pick a partition $P : a = t_1 < t_2 < \cdots < t_n = b$ of $[a,b]$ such that $\gamma \restriction_{[t_j - t_{j+1}]}$ is homotopic with fixed end points to $\gamma_{z_j \to z_{j+1}}$ in $\mathbb{C} \setminus \{z_0\}$ where $z_j = \gamma(t_j)$. Then γ is homotopic to $\gamma_{z_1 \to z_2 \to \cdots \to z_n \to z_1}$ as closed curves. Since $W_\gamma(z_0) = W_{\gamma_{z_1 \to z_2 \to \cdots \to z_n \to z_1}}(z_0)$, we may start with the assumption that γ is a closed polygonal path.

Let us now prove the result by induction. Let $z_1, \ldots, z_n$ be as above. Suppose $z_0 \in \gamma_{z_{n-1} \to z_1}$. Since $W_\gamma(z) = W_\gamma(z_0)$ for $z \in D(z_0, r)$, with r as described in Proposition 3, pick a point $z' \in D(z_0, r)$ and $z' \notin \gamma_{z_{n-1} \to z_1}$.

The induction is on the number of points $\{z_1, \ldots, z_n\}$. The case where $n = 1$ and $n = 2$ are trivial, since the polygonal path with one or two points does not form a closed curve. When $n = 3$, $\gamma = \gamma_{z_1 \to z_2 \to z_3 \to z_1}$ will be a triangle, T. Let $\hat{T}$ denote the convex hull of the triangle T. Then $\mathring{\hat{T}}$ denote the interior of $\hat{T}$. If $z_0 \in \mathring{\hat{T}}$, then by Cauchy integral formula,

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = 1.$$

If $z_0 \in \mathbb{C} \setminus \hat{T}$, then γ is null-homotopic in $\mathbb{C} \setminus \{z_0\}$ and hence

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} = 0.$$

Hence $W_\gamma(z_0)$ is an integer.

Assume the result to be proved for up to $n - 1$.

Let $\gamma = \gamma_{z_1 \to z_2 \to \cdots \to z_n \to z_1}$. Now we have

$$\begin{aligned} W_\gamma(z') &= W_{\gamma_{z_1 \to z_2 \cdots \to z_{n-1} \to z_1}}(z') + W_{\gamma_{z_1 \to z_{n-1} \to z_n \to z_1}}(z') \\ &= \frac{1}{2\pi i} \int_{\gamma_{z_1 \to z_2 \cdots \to z_{n-1} \to z_1}} \frac{dz}{z - z'} + \frac{1}{2\pi i} \int_{\gamma_{z_1 \to z_{n-1} \to z_n \to z_1}} \frac{dz}{z - z'} \end{aligned}$$

Now by the induction hypothesis, first and second terms on the R.H.S are integers. Hence $W_\gamma(z')$ is an integer which implies $W_\gamma(z_0)$ is also an integer. □

THEOREM 5 (**General Cauchy Integral Formula**). *Let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic on open set Ω and $\gamma : [a, b] \longrightarrow \Omega$ be a closed curve which is null-homotopic. Then for $z_0 \notin \gamma([a, b])$, we have*

$$f(z_0) W_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)}.$$

PROOF. By factorization theorem for holomorphic functions,

$$f(z) - f(z_0) = (z - z_0) g(z)$$

and hence,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(z_0)}{(z - z_0)} = \frac{1}{2\pi i} \int_\gamma g(z).$$

Since γ is null homotopic on Ω, by Cauchy's theorem,

$$\frac{1}{2\pi i} \int_\gamma g(z) = 0.$$

Therefore,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)} = \frac{1}{2\pi i} \int_\gamma \frac{f(z_0)}{(z - z_0)} = f(z_0) W_\gamma(z_0).$$

□