Complex Analysis

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Lecture No – 28

Problem Session

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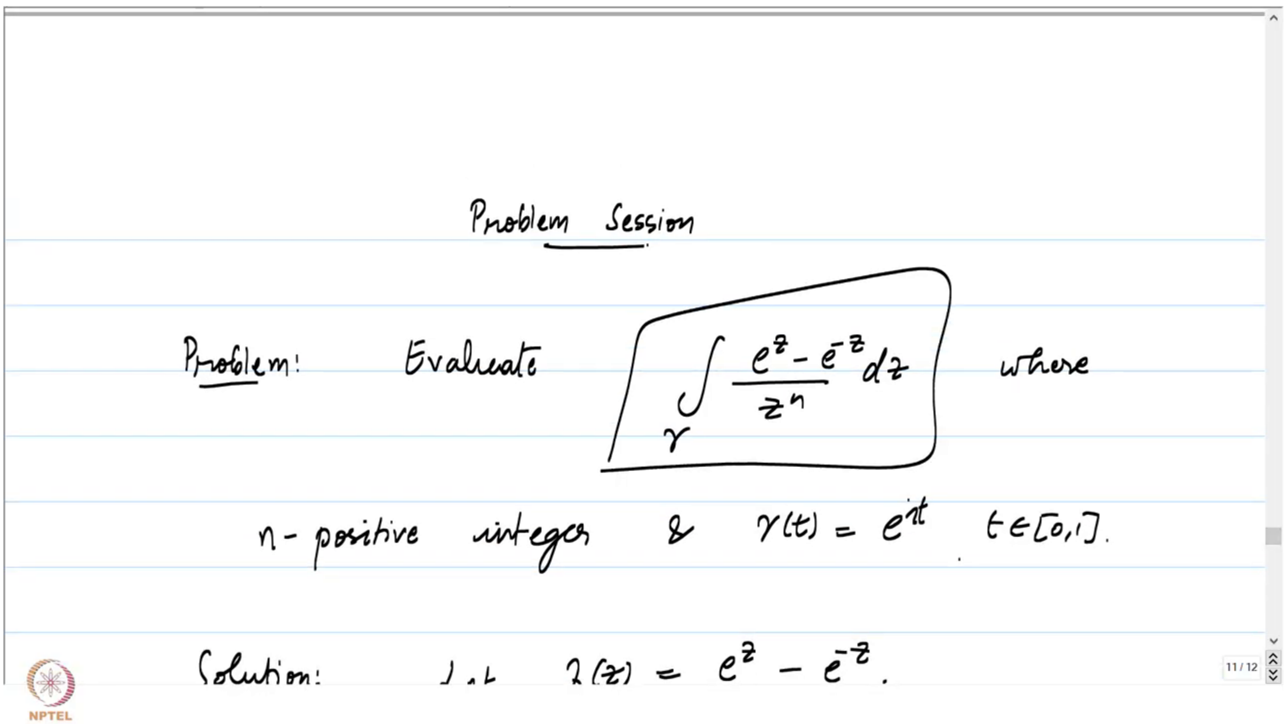

PROBLEM 1. Evaluate

$$\int_\gamma \frac{e^z - e^{-z}}{z^n} dz$$

where $n \in \mathbb{N}$ and $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.

SOLUTION 1. Let $f(z) = e^z - e^{-z}$. Then f is an entire function. By higher order Cauchy integral formula we have,

$$f^{n-1}(0) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{f(z)}{z^n} dz.$$

Hence,

$$\int_\gamma \frac{e^z - e^{-z}}{z^n} dz = \frac{2\pi i f^{n-1}(0)}{(n-1)!}.$$

Now it is an easy check to verify that,

$$f^k(z) = \begin{cases} e^z - e^{-z} & \text{if } k \text{ is even} \\ e^z + e^{-z} & \text{if } k \text{ is odd.} \end{cases}$$

Therefore,

$$\begin{aligned} \int_\gamma \frac{e^z - e^{-z}}{z^n} dz &= \frac{2\pi i f^{n-1}(0)}{(n-1)!} \\ &= \begin{cases} \dfrac{4\pi i}{(n-1)!} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

PROBLEM 2. Evaluate

$$\int_\gamma \frac{z^2+1}{z(z^2+4)} dz$$

where $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$, when $0 < r < 2$ and $r > 2$.

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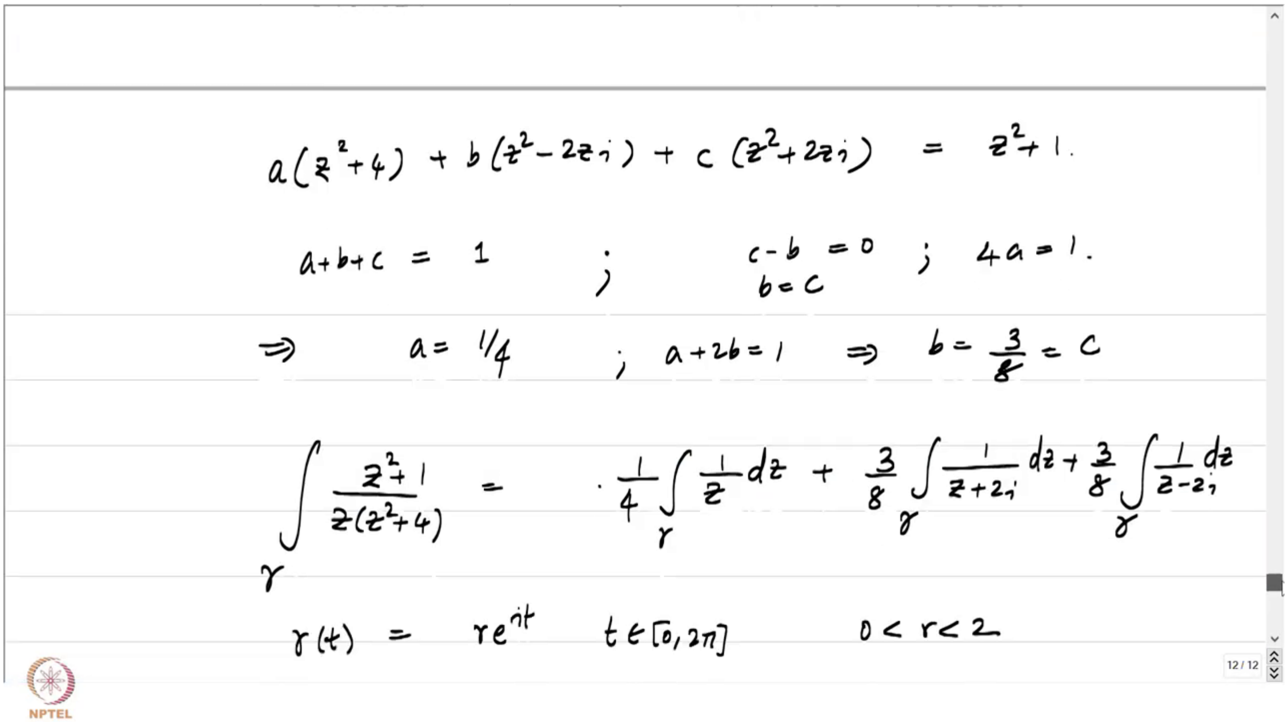

SOLUTION 2. We may spilt the integrand by using the partial fractions.

$$\frac{z^2+1}{z(z^2+4)} = \frac{a}{z} + \frac{b}{(z+2i)} + \frac{c}{z-2i}.$$

By our routine calculation, we will get the values of the coefficients a, b and c as, $a = \frac{1}{4}, b = c = \frac{3}{8}$.

Now we have,

$$\int_\gamma \frac{z^2+1}{z(z^2+4)}dz = \frac{1}{4}\int_\gamma \frac{dz}{z} + \frac{3}{8}\int_\gamma \frac{dz}{z+2i} + \frac{3}{8}\int_\gamma \frac{dz}{z-2i}.$$

First we shall consider the case when $0 < r < 2$ and we have $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$.

We know that $\frac{1}{z+2i}$ is holomorphic on $\mathbb{C} \setminus \{-2i\}$. Also, γ is null-homotopic on $\mathbb{C} \setminus \{-2i\}$. By Cauchy's theorem,

$$\int_\gamma \frac{dz}{z+2i} = 0.$$

Similarly,

$$\int_\gamma \frac{dz}{z-2i} = 0.$$

Hence, when $0 < r < 2$, we have,

$$\int_\gamma \frac{z^2+1}{z(z^2+4)}dz = \frac{1}{4}\int_\gamma \frac{dz}{z} = \frac{\pi i}{2}.$$

Now, let us consider the case when $r > 2$.

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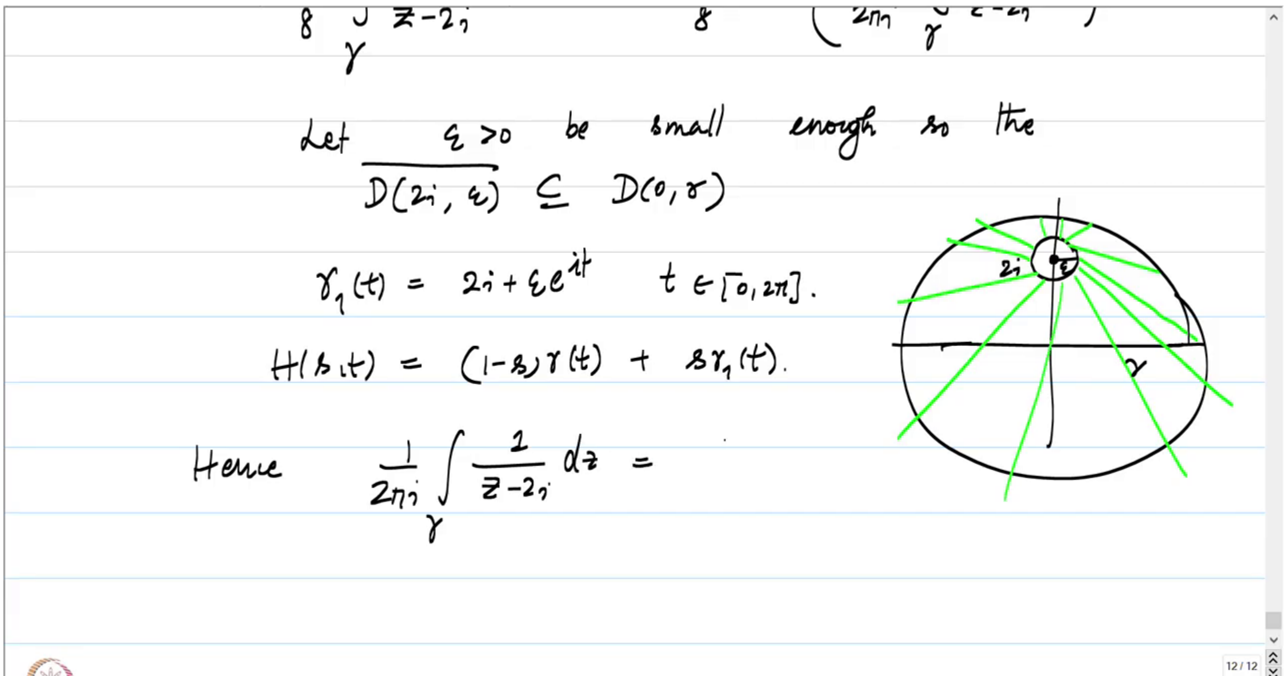

Let $\epsilon > 0$ be small enough such that $\overline{D(2i,\epsilon)} \subseteq D(0,r)$ and let $\gamma_1(t) = 2i + \epsilon e^{it}$, $t \in [0,2\pi]$. Now γ is homotopic γ_1 as closed curves by the straight line homotopy,

$$H(s,t) = (1-s)\gamma(t) + s\gamma_1(t), \qquad (s,t) \in [0,1] \times [0,2\pi].$$

Hence by Cauchy's theorem, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-2i} = \frac{1}{2\pi i}\int_{\gamma_1} \frac{dz}{z-2i}$$

and by the Cauchy integral formula,

$$\frac{3}{8}\int_\gamma \frac{dz}{z-2i} = \frac{3}{8}\cdot 2\pi i \cdot \frac{1}{2\pi i}\int_{\gamma_1}\frac{dz}{z-2i} = \frac{3\pi i}{4}.$$

Similarly, we have

$$\frac{3}{8}\int_\gamma \frac{dz}{z+2i} = \frac{3\pi i}{4}$$

and

$$\frac{1}{4}\int_\gamma \frac{dz}{z} = \frac{\pi i}{2}.$$

Hence,

$$\int_\gamma \frac{z^2+1}{z(z^2+4)}dz = \frac{\pi i}{2} + \frac{3\pi i}{2} = 2\pi i.$$

PROBLEM 3. Let f be a bounded entire function and $a, b \in \mathbb{C}$ be two distinct complex numbers. Let $R > \max(|a|, |b|)$. Then evaluate,

$$I_R = \int_\gamma \frac{f(z)}{(z-a)(z-b)}dz$$

where $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$. Also evaluate $\lim_{R\to\infty} I_R$.

SOLUTION 3.

$$\begin{aligned}
I_R &= \int_\gamma \frac{f(z)}{(z-a)(z-b)}dz \\
&= \int_\gamma \frac{1}{(a-b)}\left(\frac{f(z)}{(z-a)} - \frac{f(z)}{(z-b)}\right)dz \\
&= \frac{1}{(a-b)}\int_\gamma \frac{f(z)}{z-a}dz - \frac{1}{(a-b)}\int_\gamma \frac{f(z)}{z-b}dz \\
&= \frac{2\pi i}{(a-b)}f(a) - \frac{2\pi i}{(a-b)}f(b) \\
I_R &= 2\pi i\left(\frac{f(a)-f(b)}{a-b}\right).
\end{aligned}$$

Notice that f is bounded, thus there exists an $M > 0$ such that $|f(z)| < M$ for every $z \in \mathbb{C}$. Since, $|z-a| \geq |z| - |a| \geq R - |a|$ and $|z-b| \geq R - |b|$,

$$\left|\frac{f(z)}{(z-a)(z-b)}\right| \leq \frac{M}{(R-|a|)(R-|b|)}$$

and hence

$$|I_R| \leq \frac{M}{(R-|a|)(R-|b|)} 2\pi R.$$

Now,

$$\lim_{R\to\infty} |I_R| \leq \lim_{R\to\infty} \frac{M}{(R-|a|)(R-|b|)} 2\pi R = 0.$$

Hence $\lim_{R\to\infty} I_R = 0 \implies 2\pi i \left(\frac{f(a)-f(b)}{a-b}\right) = 0 \implies f(a) = f(b)$, i.e., f is a constant.

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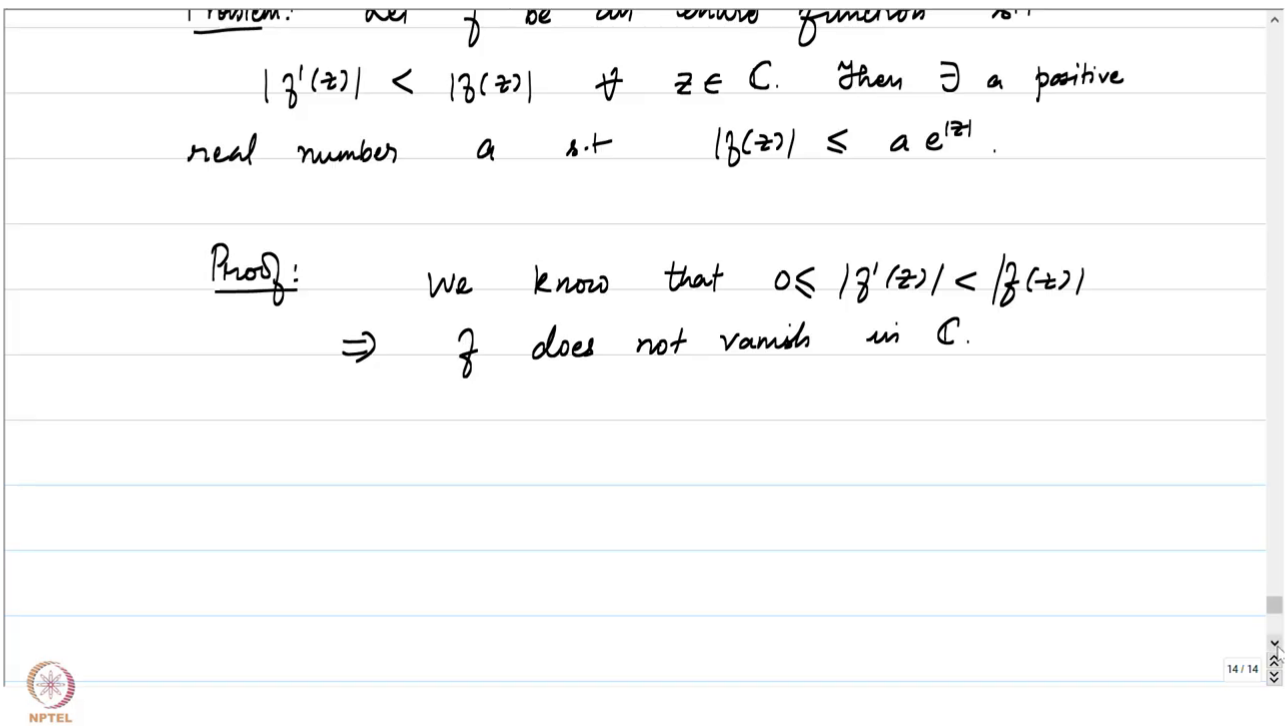

PROBLEM 4. Let f be an entire function such that $|f'(z)| < |f(z)|$ for every $z \in \mathbb{C}$. Then there exists a positive real number a such that

$$|f(z)| \leq ae^{|z|}.$$

SOLUTION 4. Since $0 \le |f'(z)| < |f(z)|$, we have f is does not vanish on $\mathbb{C}$. Hence $\dfrac{1}{f}$ is an entire function. Let $g(z) = \dfrac{f'(z)}{f(z)}$. Then g is holomorphic on $\mathbb{C}$ and also $|g(z)| < 1$ for every $z \in \mathbb{C}$. By Liouville's theorem, $g \equiv c$ for some $c \in \mathbb{C}$, i.e., $f'(z) = cf(z)$.

We know that f is an entire function, hence there exists a power series expansion,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

in $\mathbb{C}$. Also we know that,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Since $f'(z) = cf(z)$, we have

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} c a_n z^n.$$

Hence

$$ca_0 = a_1,\ ca_1 = 2a_2, \ldots, ca_k = (k+1)!a_{k+1} \implies a_k = \frac{c^k a_0}{k!}.$$

Then we have,

$$f(z) = \sum_{n=0}^{\infty} \frac{a_0 c^n z^n}{n!} = a_0 \sum_{n=0}^{\infty} \frac{(cz)^n}{n!} = a_0 \cdot e^{cz}.$$

PROBLEM 5. Suppose f is an entire function such that $|f(z)| \le a + b|z|^k$ for every $z \in \mathbb{C}$, where $a, b, k \in \mathbb{N}$. Then f is a polynomial.

SOLUTION 5. Let $z_0 \in \mathbb{C}$ and $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ be a curve given by $\gamma(t) = z_0 + Re^{it}$.

Now by the higher order Cauchy integral formula,

$$\left| f^{(k+1)}(z_0) \right| = \left| \frac{(k+1)!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+2}} \right|. \tag{1}$$

Since $|f(z)| \le a + b|z|^k$, there must exists positive integers a' and b' such that $|f(z)| \le a' + b'|z - z_0|^k$ for every $z \in \mathbb{C}$.

Then in (1), we have

$$\left|\frac{(k+1)!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{k+2}}\right| \le \frac{(k+1)!}{2\pi}\frac{(a'+b'R^k)}{R^{k+2}}2\pi R$$
$$= (k+1)!\left(\frac{a'}{R^{k+1}}+\frac{b'}{R}\right).$$

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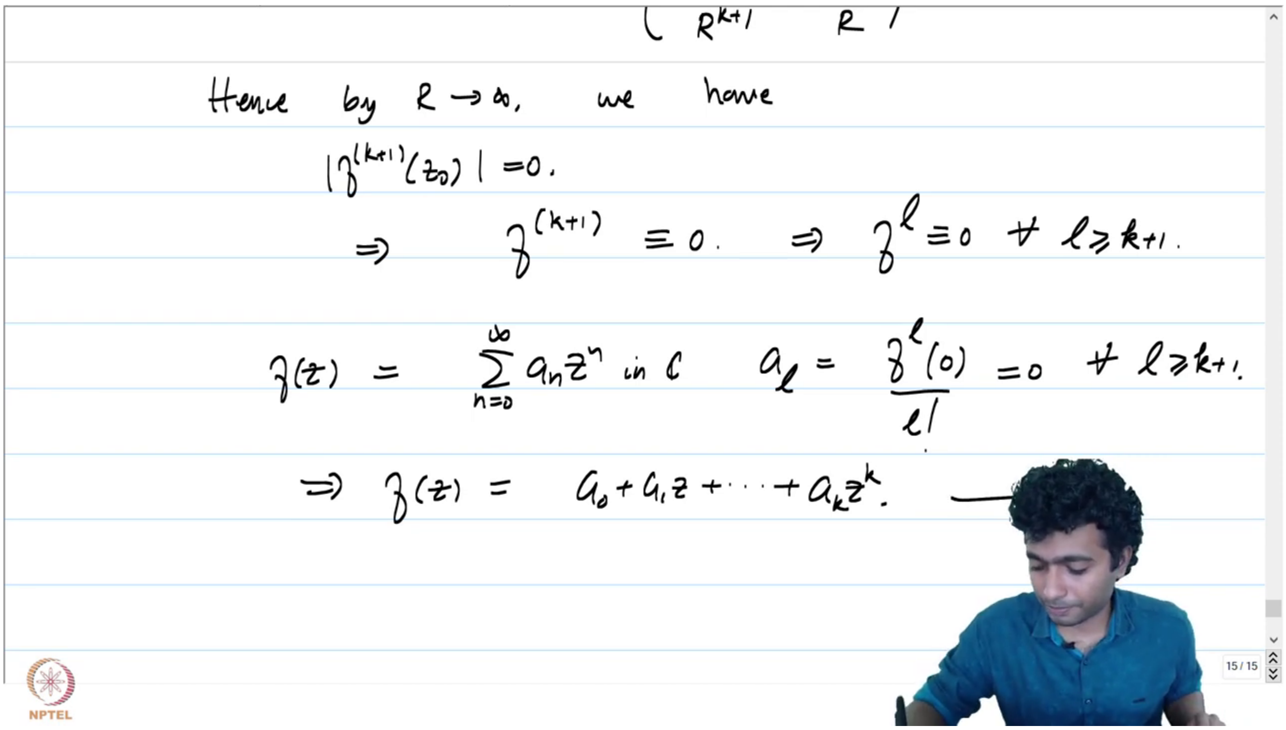

Hence, if $R \to \infty$, we have $|f^{k+1}(z_0)| = 0 \implies f^{k+1}(z) = 0$ for every $z \in \mathbb{C}$. Thus $f^\ell \equiv 0$ for every $\ell \geq k+1$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{k} a_k z^k$$

which is a polynomial.

PROBLEM 6. Does there exists a holomorphic function on $\mathbb{D}$, the unit disc, such that $f(z_n) = 0$ where $\{z_n\}$ is a countable set in $\mathbb{D}$ consisting of distinct points.

SOLUTION 6. Consider $z_n = 1 - \dfrac{1}{n\pi}$ and the function defined by $f(z) = \sin\left(\dfrac{1}{1-z}\right)$. Then,

$$f(z_n) = \sin\left(\frac{1}{1-\left(1-\frac{1}{n\pi}\right)}\right) = \sin(n\pi) = 0.$$