Complex Analysis

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Lecture No – 27

Further Consequences of Cauchy Integral Formula

In the last lecture we established inequalities on the higher order derivatives of a holomorphic function defined on a domain Ω. They were called the Cauchy estimates or the Cauchy inequalities. And we will be using the Cauchy inequalities to establish now that a non-constant holomorphic function defined on the complex plane cannot be bounded. This result is called Liouville's theorem.

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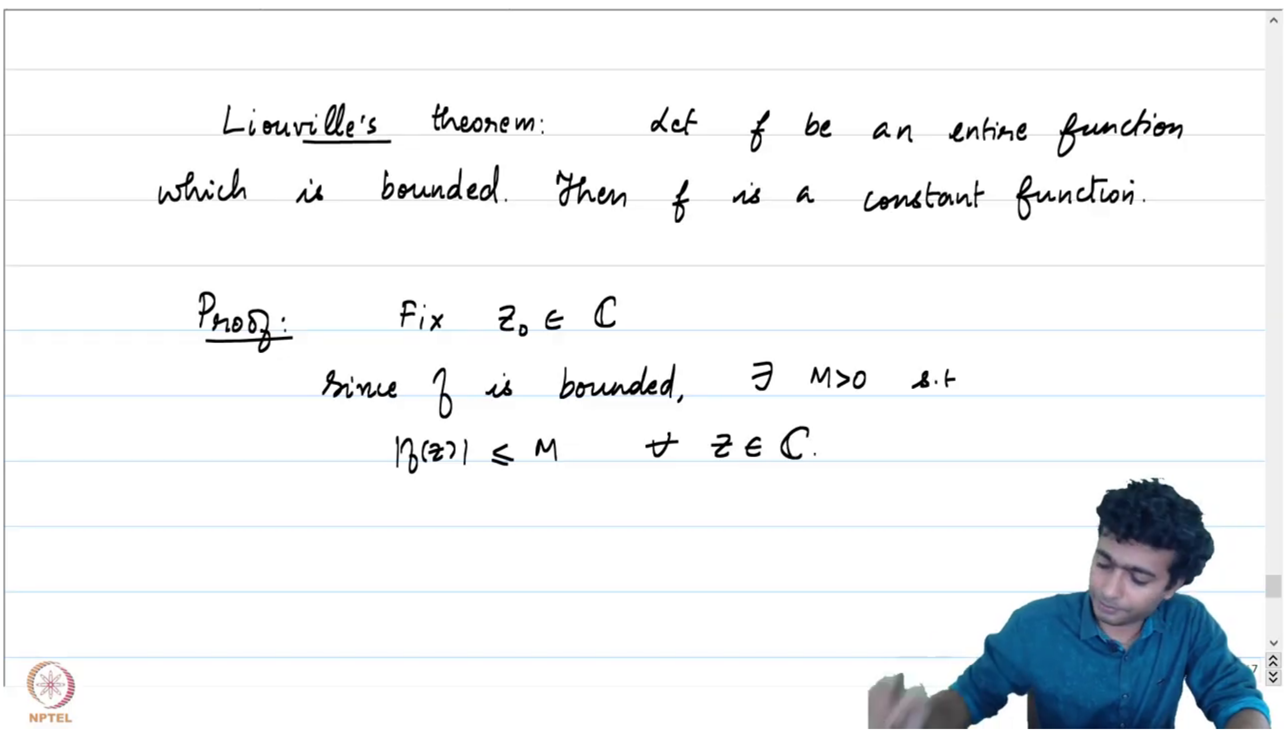

THEOREM 1 (Liouville's Theorem). *Let f be an entire function which is bounded. Then f is a constant function.*

PROOF. Since f is bounded, there exists $M > 0$ such that $f(z) \leq M$ for every $z \in \mathbb{C}$. Fix $z_0 \in \mathbb{C}$. For $r > 0$, let $\sigma_r(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then

$$|f(z)| < M \qquad \forall\, z \in \sigma_r([0,2\pi]) \text{ and } \forall\, r > 0.$$

By Cauchy inequality,

$$|f'(z_0)| \leq \frac{M}{r}.$$

Taking the limit as $r \to \infty$, we get

$$f'(z_0) = 0.$$

Since the point z_0 was arbitrary, we have $f' \equiv 0$. By fundamental theorem of calculus, f is a constant function. □

So bounded entire functions are necessarily constants. This is actually in stark contrast to what can be seen in the Real analysis setting. If we were to look at the real line, we have functions like sine which is real analytic and bounded by 1. This cannot happen in the Complex setting. The moment we have an entire function that is bounded, it is forced to be a constant function. Sine function certainly has an extension to the complex plane. The reader should convince themselves that sine function away from the real line is not bounded.

THEOREM 2 (Fundamental Theorem of Algebra). *Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a non-constant polynomial with $a_j \in \mathbb{C}\ \forall\, 0 \leq j \leq n$ such that $a_n \neq 0$. Then there exist $z_1, \ldots, z_n$ (not necessarily distinct) such that $p(z) = a_n(z - z_1)\ldots(z - z_n)$.*

PROOF. We shall prove this theorem by using induction.

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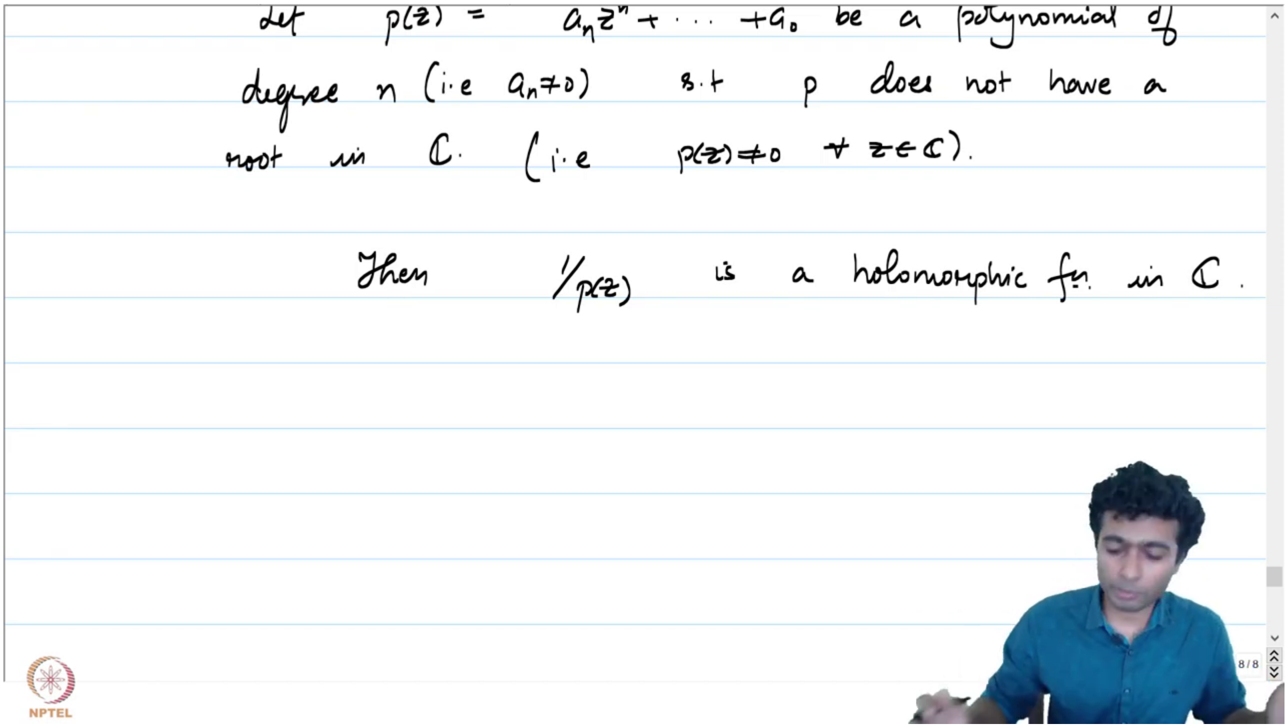

For $n = 1$, the theorem follows directly. Assume that the result has been established up to $n-1$.

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n such that p does not have a root in $\mathbb{C}$, i.e., $p(z) \neq 0$ for every $z \in \mathbb{C}$. Then $\dfrac{1}{p(z)}$ is a holomorphic function in $\mathbb{C}$.

For $|z| > R > 1$,

$$
\begin{aligned}
|p(z)| &= \left|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0\right| \\
&\geq |a_n||z|^n - \left|a_{n-1} z^{n-1} + \cdots + a_0\right| \\
&\geq |z|^n \left(|a_n| - \frac{1}{|z|}\left|a_{n-1} + \cdots + \frac{a_0}{z^{n-1}}\right|\right).
\end{aligned}
$$

Now let us focus on the second term in the RHS. For R large enough, we have

$$\begin{aligned}\frac{1}{|z|}\left|a_{n-1}+\cdots+\frac{a_0}{z^{n-1}}\right| &\leq \frac{1}{|z|}\left(|a_{n-1}|+\cdots+\frac{|a_0|}{|z^{n-1}|}\right)\\ &\leq \frac{1}{|z|}\left(|a_{n-1}|+\cdots+|a_0|\right)\\ &< |a_n|.\end{aligned}$$

Hence

$$|a_n| - \frac{1}{|z|}\left|a_{n-1}+\cdots+\frac{a_0}{z^{n-1}}\right| > M' > 0.$$

That is, if we choose $R > 1$ large enough, we have

$$|p(z)| > |z|^n M' > R^n M'.$$

Also, given $M > 1$, there exists $R > 1$ such that $|p(z)| > M$ or $\left|\frac{1}{p(z)}\right| < M$ for every z with $|z| > R$.

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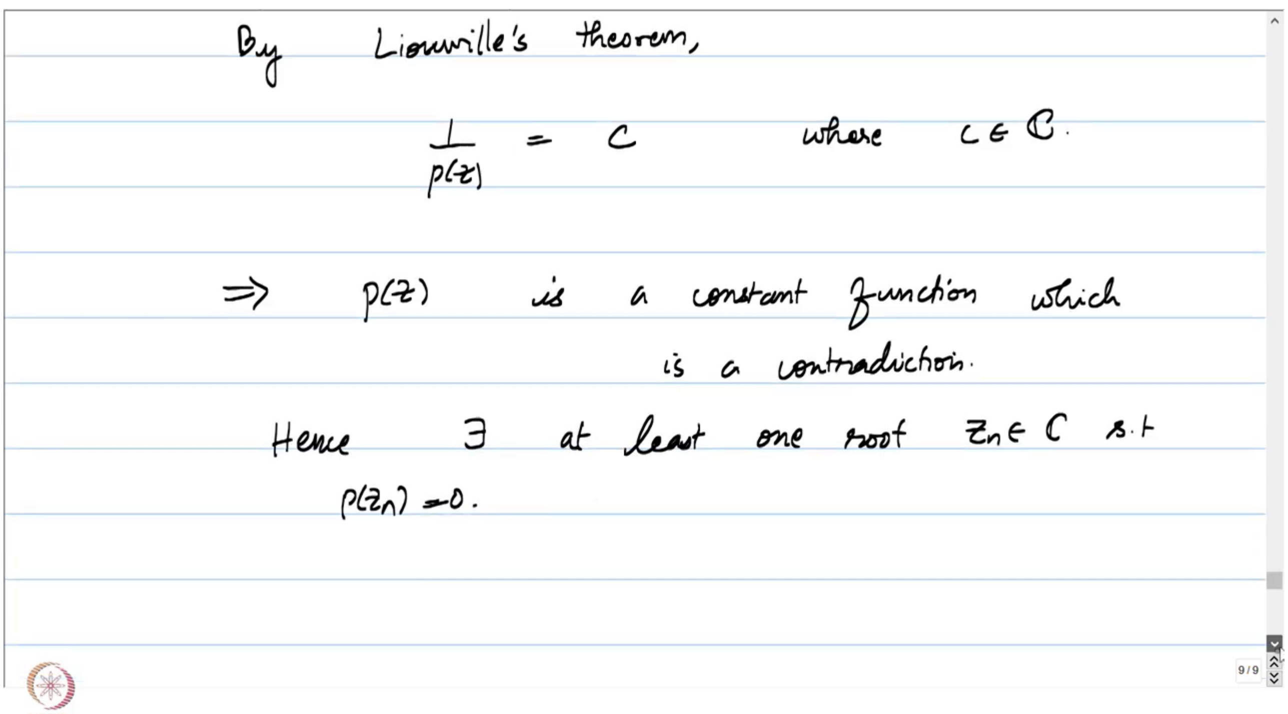

Since $\overline{D(0,R)}$ is compact and $\dfrac{1}{p(z)}$ is continuous, $\left(\dfrac{1}{p}\right)\left(\overline{D(0,R)}\right)$ is compact. Hence,

$$\left|\frac{1}{p(z)}\right| < M_1 \qquad \forall\, z \in \overline{D(0,R)}.$$

Let $M_2 = \max\{M, M_1\}$. Then we have,

$$\left|\frac{1}{p(z)}\right| < M_2 \qquad \forall\, z \in \mathbb{C}.$$

By Liouville's theorem, $\dfrac{1}{p(z)} = c$, where $c \in \mathbb{C}$ and therefore $p(z)$ is a constant function which is a contradiction.

Hence there exists at least one root $z_n \in \mathbb{C}$ such that $p(z_n) = 0$. Then we have, $p(z) = q(z)(z - z_n)$, where $q(z) = a_n z^{n-1} + b_{n-2} z^{n-2} + \cdots + b_0$. By induction,

$$q(z) = a_n(z - z_1)\ldots(z - z_{n-1}).$$

Hence, $p(z) = a_n(z - z_1)\ldots(z - z_{n-1})(z - z_n)$. □

We will now prove a form of converse to the Cauchy's theorem. Cauchy's theorem stated that, if you have a function f which is holomorphic on a given domain Ω and if you have a closed path γ which is null homotopic, then $\int_\gamma f = 0$. A form of converse would be to demand that, if $\int_\gamma f = 0$ for any closed curve γ, then our function f is holomorphic. So that is going to be the type of statement we will be proving. This theorem is called Morera's theorem.

THEOREM 3 (Morera's Theorem). *Let $\Omega \subseteq \mathbb{C}$ be an open set and $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function such that*

$$\int_\gamma f(z)\,dz = 0$$

for every closed polygonal path γ in Ω. Then f is holomorphic on Ω.

PROOF. Let $z_0 \in \Omega$ and $r > 0$ be such that $\overline{D(z_0, r)} \subseteq \Omega$.

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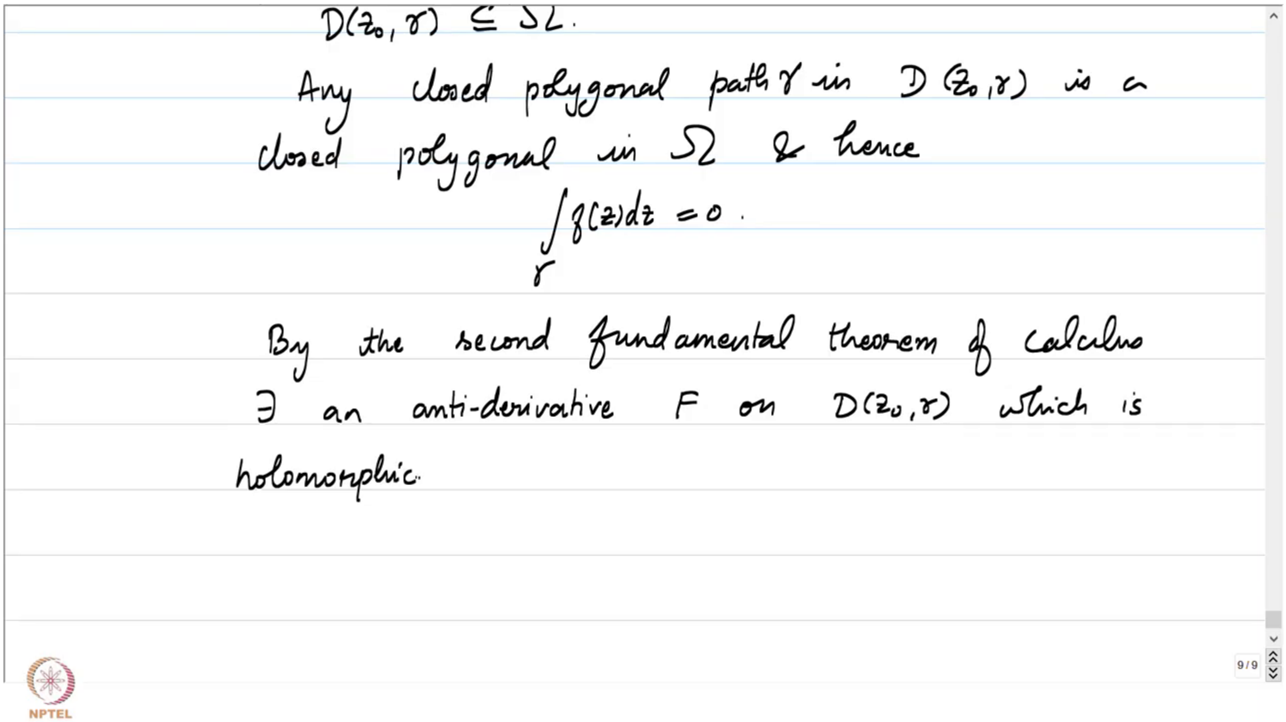

Now any closed polygonal path γ in $D(z_0, r)$ is a closed polygonal path in Ω and hence

$$\int_{\gamma} f(z)\,dz = 0.$$

By second fundamental theorem of calculus, there exists an anti-derivative F on $D(z_0, r)$ which is holomorphic. Since F is complex analytic, F' is holomorphic on $D(z_0, r)$, i.e., f is holomorphic on $D(z_0, r)$. Hence f is holomorphic on Ω. $\square$

Morera's theorem help us to conclude, if a sequence of holomorphic functions converges uniformly on compact sets to a function, then the limit is also holomorphic.

THEOREM 4. *Let $\Omega \subseteq \mathbb{C}$ be open and $f_n : \Omega \longrightarrow \mathbb{C}$ be holomorphic on Ω for each $n \in \mathbb{N}$ such that f_n converges uniformly on compact sets to a function f. Then f is holomorphic.*

PROOF. Let $z_0 \in \Omega$ and $r > 0$ be such that $\overline{D(z_0, r)} \subseteq \Omega$. Since f_n converges to f uniformly on compact sets in Ω, we have f is continuous on Ω. In particular, f_n converge to f uniformly on compact sets in $D(z_0, r)$. Let $\sigma : [0,1] \longrightarrow \mathbb{C}$ be a closed polygonal path in $D(z_0, r)$. Since $\sigma([0,1])$ is compact, f_n converges to f uniformly on $\sigma([0,1])$. Then we have,

$$\lim_{n\to\infty} \int_\sigma f_n(z)dz = \int_\sigma \lim_{n\to\infty} f_n(z)dz = \int_\sigma f(z)dz.$$

But

$$\int_\sigma f_n(z)dz = 0 \qquad \forall\, n \in \mathbb{N},$$

hence

$$\int_\sigma f(z)dz = 0.$$

By Morera's theorem, we have f is holomorphic on Ω. □

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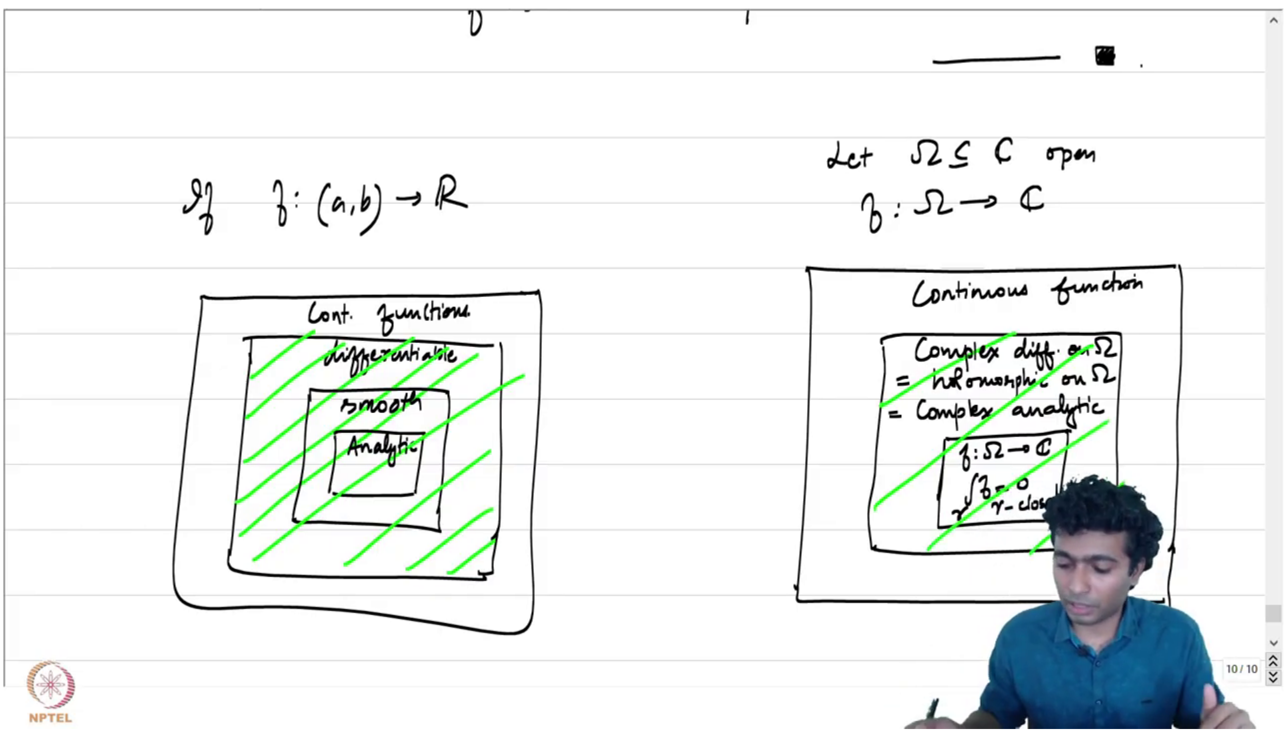

Before we conclude, let us summarize what are the things that we have concluded about complex valued functions on domains in $\mathbb{C}$ till now. Let f be a function such that $f : (a,b) \longrightarrow \mathbb{R}$. If f is continuous then it belongs to a broader class of functions. If we put more and more regularity to the function f, the size of set to which it belong will become 'small'. That is, f can be $k-$times differentiable but it is not $k+1-$times differentiable. Then we can put f in a subclass of continuous functions called differentiable functions. If f has a regularity of $\mathscr{C}^\infty-$smooth, then it will fall into the subclass of differentiable functions called the smooth functions. We have a further subclass called analytic functions. There are functions which are smooth but not analytic. For example, consider the function,

$$f(x) = \begin{cases} e^{\frac{-1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then f is smooth but it is not analytic. The reader should verify the details.

But if we look into the functions on an open subset Ω of $\mathbb{C}$, we however do not have as many classifications as we have in the real analysis setting. There will certainly be the broad class of continuous functions, but the moment we go down to a subclass of complex differentiable functions on Ω that is going to be the same as complex analytic functions or holomorphic functions. But this is the same as complex analytic which we have proved.

Of course, there is one more interesting class of functions; the set of all those functions, $f : \Omega \longrightarrow \mathbb{C}$ such that $\int_\gamma f = 0$ where γ is a closed curve. Now, notice that this is actually a special class of functions because if we consider $\Omega = \mathbb{C} \setminus \{0\}$ and $f(z) = \frac{1}{z}$. Then f is holomorphic on Ω but $\int_\sigma f(z) = 2\pi i \neq 0$, where $\sigma(t) = e^{it}$ for $t \in [0, 2\pi]$.