Complex Analysis

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Lecture No – 26

Principle of Analytic Continuation and Cauchy Estimates

We saw in the last lecture that holomorphic functions can be identified with complex analytic functions. We will now look at one more corollary to the theorem, which states that a holomorphic function can be written as a power series on sufficiently small disc in the domain of definition, called the factor theorem for analytic functions.

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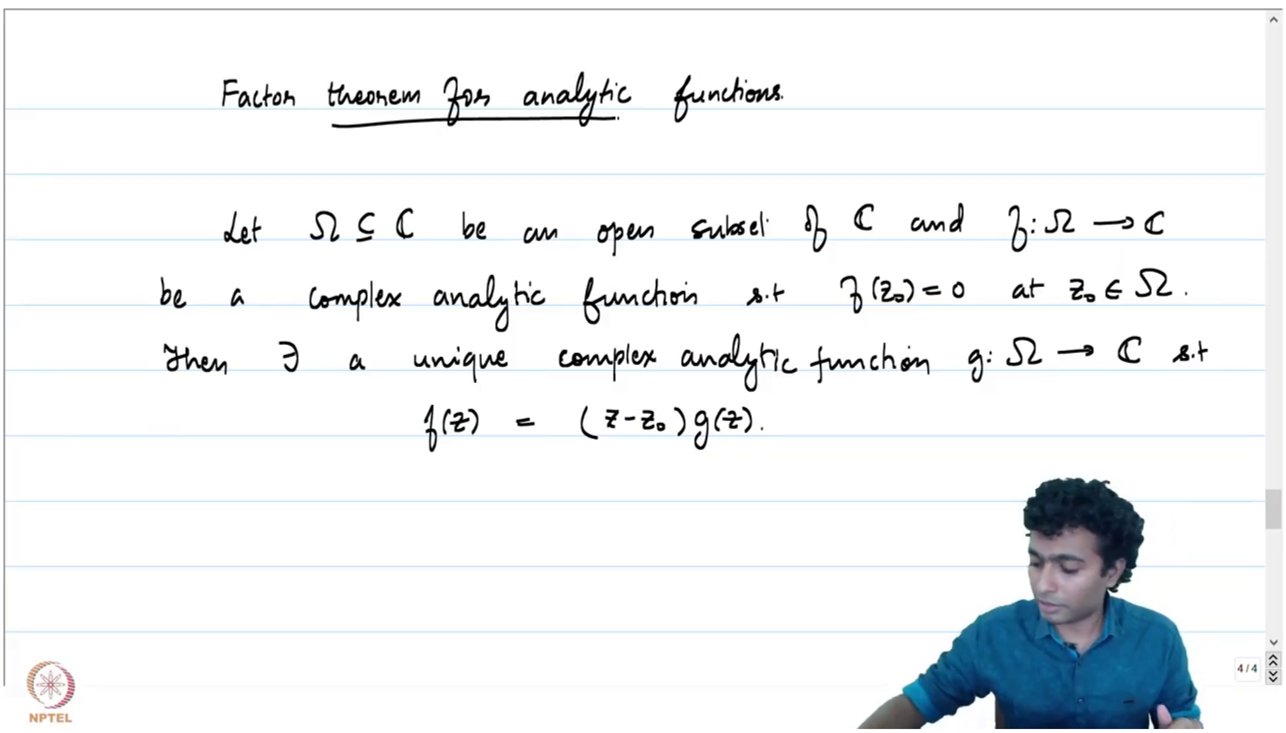

THEOREM 1 (Factor Theorem for Analytic Functions). *Let $\Omega \subseteq \mathbb{C}$ be an open set and $f : \Omega \longrightarrow \mathbb{C}$ be a complex analytic function such that $f(z_0) = 0$ at $z_0 \in \Omega$. Then there exists a unique complex analytic function $g : \Omega \longrightarrow \mathbb{C}$ such that $f(z) = (z - z_0)g(z)$.*

PROOF. Since $\dfrac{1}{z-z_0}$ is holomorphic on $\Omega \setminus \{z_0\}$ we have $\dfrac{f(z)}{z-z_0}$, is holomorphic on $\Omega \setminus \{z_0\}$.

By the hypothesis, f is complex analytic on $D(z_0, r)$, where $\overline{D(z_0, r)} \subseteq \Omega$, we have the following power series expansion,

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

For $z \in D(z_0, r)$, define

$$g(z) := \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n.$$

It is left as an exercise to the reader to verify that g converges in $D(z_0, r)$.

On $D(z_0, r)$, we have

$$(z - z_0) g(z) = (z - z_0) \left(\sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \right) = \sum_{n=1}^{\infty} a_n (z - z_0)^n = f(z).$$

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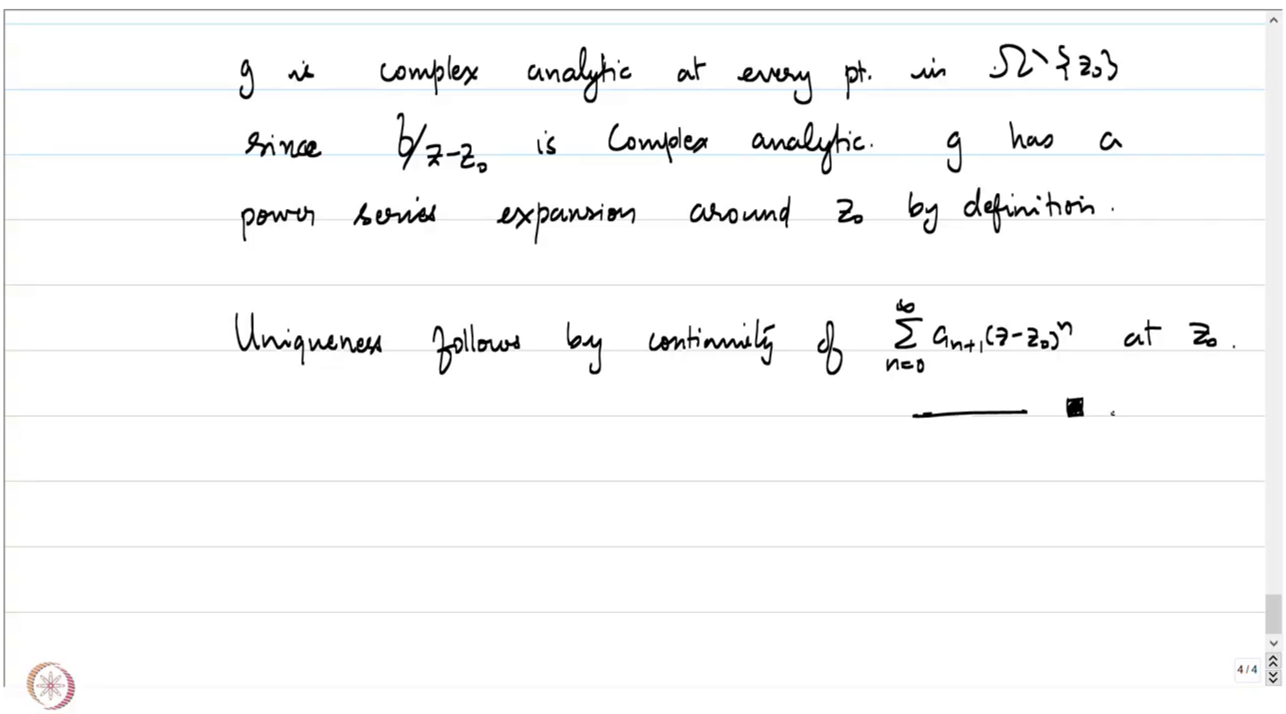

Hence on $D(z_0,r)$, $f(z)=(z-z_0)g(z)$. Then, on $D(z_0,r)\setminus\{z_0\}$, $g(z)$ coincides with $\dfrac{f(z)}{z-z_0}$.

Since $\dfrac{f(z)}{z-z_0}$ is complex analytic on $\Omega\setminus\{z_0\}$, g is complex analytic at every point in $\Omega\setminus\{z_0\}$. Also g has a power series expansion around z_0 by the definition.

Uniqueness of the function g follows from the continuity of $\sum\limits_{n=0}^{\infty} a_{n+1}(z-z_0)^n$ at z_0. □

Let us prove one more consequence. This consequence is classically called as the principle of analytic continuation.

THEOREM 2 (Principle of Analytic Continuation). *Let $\Omega \subseteq \mathbb{C}$ be an open connected subset and $f, g : \Omega \longrightarrow \mathbb{C}$ be complex analytic on Ω. Suppose f and g agree on a non-empty open subset of Ω. Then $f \equiv g$ on Ω.*

PROOF. Let us define a subset E of Ω to be

$$E := \left\{z \in \Omega : f^n(z) = g^n(z) \quad \forall\, n \in \mathbb{N}\right\}.$$

Since f and g agrees on an open set, E is non-empty. If we manage to prove that E is both open and closed, then by the connectedness of Ω implies that $E = \Omega$.

We shall prove first that E is closed. For $n \in \mathbb{N}$, let $E_n = \{z \in \Omega : f^n(z) = g^n(z)\}$. Since both f^n and g^n are continuous, E_n is closed for each $n \in \mathbb{N}$. Further $E = \bigcap\limits_{n\in\mathbb{N}} E_n$, which is closed.

Next we shall prove that E is open. Let $z_0 \in E$. Then $f^n(z_0) = g^n(z_0) \quad \forall\, n \in \mathbb{N}$. Hence f and g have the same power series expansion in $D(z_0,r)$, where $\overline{D(z_0,r)} \subseteq \Omega$. Then for $w \in D(z_0,r)$, we have $f^n(w) = g^n(w) \quad \forall\, n \in \mathbb{N}$. Hence $D(z_0,r) \subseteq E$.

Hence E is both open and closed. □

We can prove that the zero sets of a non-trivial holomorphic function will always be isolated. In some sense, we are proving a stronger version of what we have just proved. If two functions defined on a connected open set agree on a subset that has a limit point, we will be able to show that those two functions are identical in the domain of definition. This is stronger than the principle of analytic continuation in the sense that here we are not demanding that these two functions should agree on an open set for being identical but a condition weaker than that. This theorem is also sometimes called the identity theorem.

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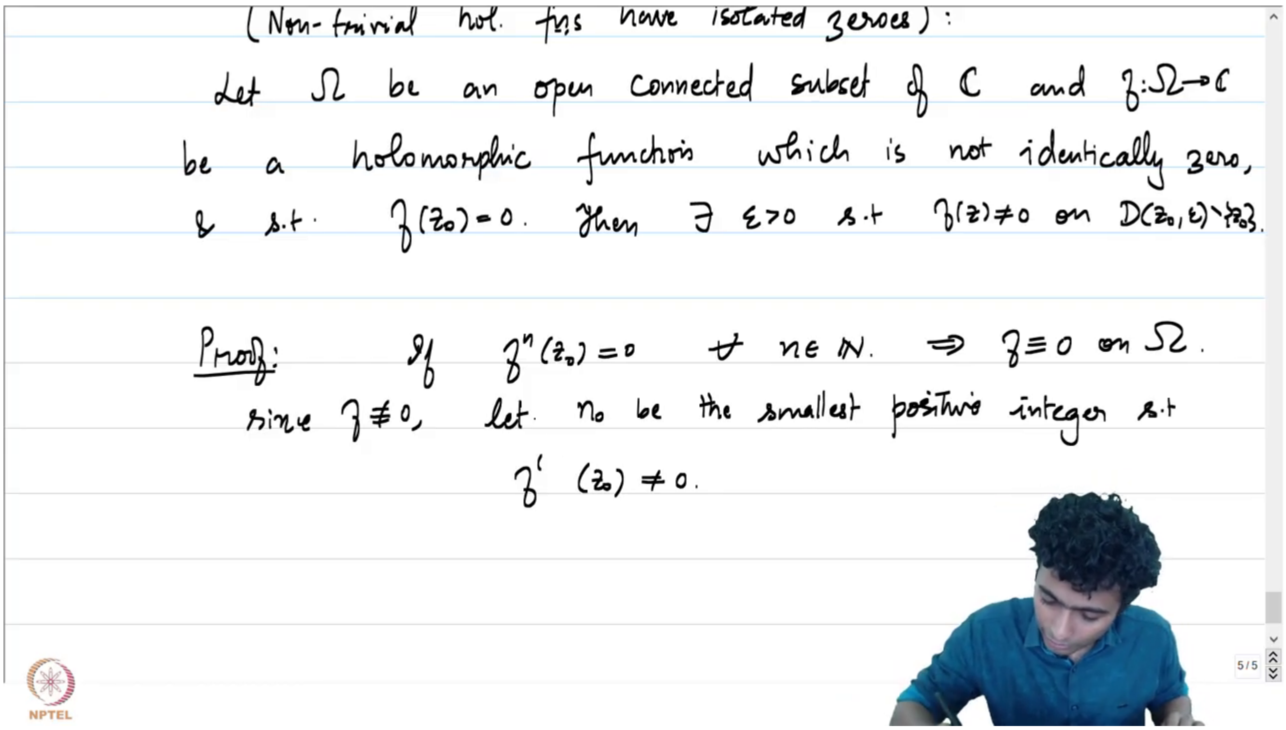

THEOREM 3 (Identity Theorem). *Let* $\Omega \subseteq \mathbb{C}$ *be an open connected set and* $f : \Omega \longrightarrow \mathbb{C}$ *be a holomorphic function which is not identically zero and such that* $f(z_0) = 0$. *Then there exists* $\epsilon > 0$ *such that* $f(z) \neq 0$ *on* $D(z_0, \epsilon) \setminus \{z_0\}$.

PROOF. If $f^n(z_0) = 0$ for every $n \in \mathbb{N}$, then $f \equiv 0$ on Ω by Theorem 2. Since by the hypothesis $f \not\equiv 0$, $\exists\, m \in \mathbb{N}$ such that $f^m(z_0) \neq 0$. Let n_0 be the smallest positive integer such that $f^{n_0}(z_0) \neq 0$.

By the choice of n_0, we have $f^k(z_0) = 0$ for every $k < n_0$. Applying Theorem 1 iteratively, we have

$$f(z) = (z - z_0)^{n_0} g(z).$$

If $g(z_0) = 0$, then $f^{n_0}(z_0) = 0$ which is a contradiction. Hence $g(z_0) \neq 0$. By the continuity of g, there exists a $\epsilon > 0$ such that $g(z) \neq 0$ for every $z \in D(z_0, \epsilon)$. Also $(z - z_0)$ does not vanish on $D(z_0, r) \setminus \{z_0\}$. Hence $f(z) = (z - z_0)^{n_0} g(z)$ does not vanish on $D(z_0, r) \setminus \{z_0\}$. $\square$

EXAMPLE 4. Consider $f(z) = \sin^2(z) + \cos^2(z)$. We know that $\sin^2(x) + \cos^2(x) = 1$ for every $x \in \mathbb{R}$. Since $\mathbb{R} \times \{0\}$ is a closed subset of $\mathbb{C}$, by Theorem 3, we have $f(z) \equiv 1$. That is $\sin^2(z) + \cos^2(z) = 1$ on $\mathbb{C}$.

THEOREM 5 (Higher Order Cauchy Integral Formula). *Let $f : \Omega \longrightarrow \mathbb{C}$ be complex analytic on an open set $\Omega \subseteq \mathbb{C}$ an $z_0 \in \Omega$ with $\overline{D(z_0, r)} \subseteq \Omega$. Let γ be a closed curve in $\Omega \setminus \{z_0\}$ homotopic as closed curves to a reparametrization of γ_1 where $\gamma_1(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then for each $n \in \mathbb{N}$*

$$f^n(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{n!}{2\pi i} \int_{\gamma_1} \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

PROOF. Since f is complex analytic at z_0 and the coefficients a_n in the power series expansion of f is given by,

$$a_n = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

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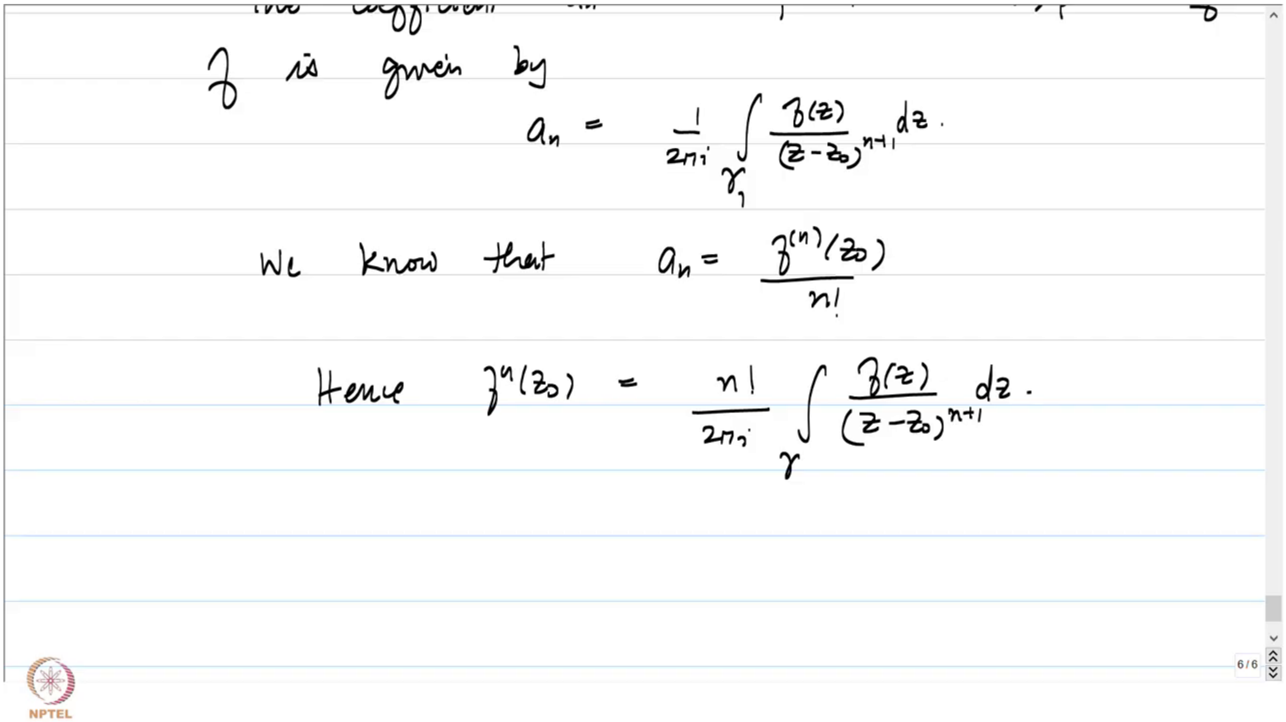

We know that $a_n = \dfrac{f^n(z_0)}{n!}$. Hence,

$$f^n(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

$\square$

And as a consequence, we directly have the Cauchy estimates using the higher order Cauchy integral formula. By this we will be able to place bounds on the derivatives of f in Ω.

COROLLARY 6 (Cauchy Estimates). *Let $\Omega \subseteq \mathbb{C}$ be an open subset and $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function. Suppose $z_0 \in \Omega$ be such that $\overline{D(z_0, r)} \subseteq \Omega$ for some $r > 0$. Let $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Suppose $|f(z)| \le M$ for every $z \in \gamma([0, 2\pi])$. Then for every $n \in \mathbb{N}$,*

$$f^n(z_0) \le M\frac{n!}{r^n}.$$

PROOF. By Theorem 5,

$$\begin{aligned}\left|f^n(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \\ &\leq \frac{n!}{2\pi}\left(\frac{M}{r^{n+1}}\right)2\pi r = M\frac{n!}{r^n}.\end{aligned}$$

□