Complex Analysis

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Lecture No – 25

Cauchy Integral Formula

We have stated and proven Cauchy's theorem; let us realize some of its consequences. As noted earlier, Cauchy's theorem is one of the most fundamental and one of the most powerful theorems in complex analysis. We will do justice to this claim by proving some very beautiful and striking consequences: the first one among them being the Cauchy integral formula. Cauchy integral formula is a powerful theorem in itself and many times it forms that tool for application of Cauchy's theorem to conclude various things about holomorphic functions.

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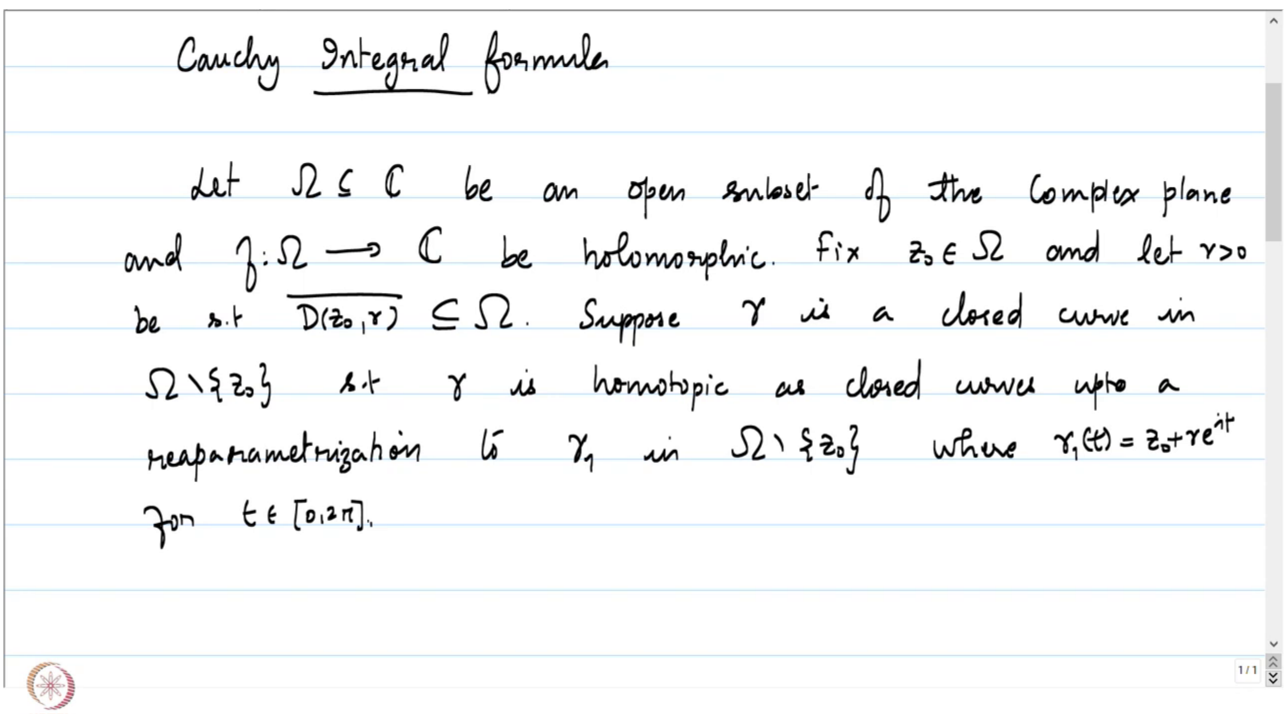

THEOREM 1 (**Cauchy Integral Formula**). *Let $\Omega \subseteq \mathbb{C}$ be an open subset of the complex plane and $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Fix $z_0 \in \Omega$ and let $r > 0$ be such that $\overline{D(z_0, r)} \subseteq \Omega$. Suppose γ is a closed curve in $\Omega \setminus \{z_0\}$ such that γ is homotopic as closed curve up to a reparametrization to γ_1 in $\Omega \setminus \{z_0\}$ where $\gamma_1(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

PROOF. Since f is holomorphic on Ω, we have $\dfrac{f(z)}{(z - z_0)}$ is holomorphic on $\Omega \setminus \{z_0\}$ and γ is homotopic as closed curve to γ_1 in $\Omega \setminus \{z_0\}$, by Cauchy's theorem,

$$\int_\gamma \frac{f(z)}{(z - z_0)} dz = \int_{\gamma_1} \frac{f(z)}{(z - z_0)} dz.$$

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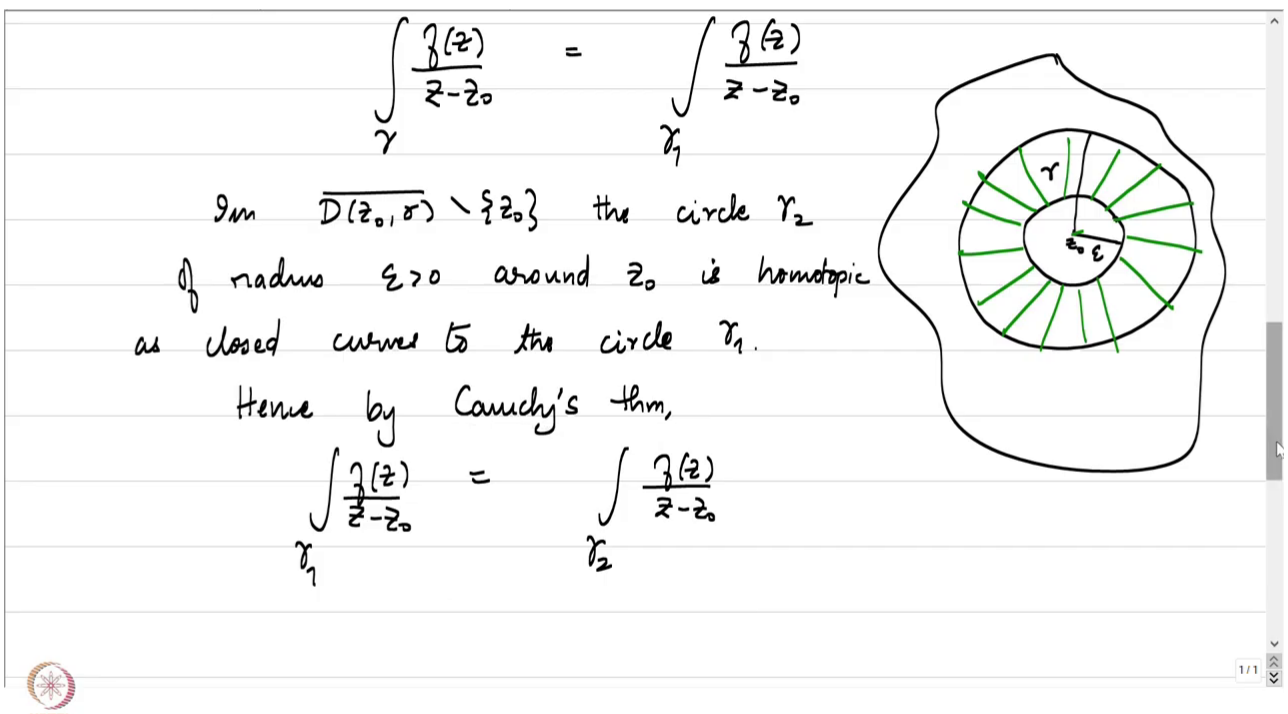

In $\overline{D(z_0,r)} \setminus \{z_0\}$, the circle, $\gamma_2(t) = z_0 + \epsilon e^{it}$ for $t \in [0,2\pi]$, of radius $\epsilon > 0$ around z_0 is homotopic as closed curves to the circle γ_1. Hence by Cauchy's theorem,

$$\int_\gamma \frac{f(z)}{(z-z_0)} dz = \int_{\gamma_1} \frac{f(z)}{(z-z_0)} dz = \int_{\gamma_2} \frac{f(z)}{(z-z_0)} dz.$$

Since f is complex differentiable at z_0, $\exists \epsilon' > 0$ such that for $\epsilon < \epsilon'$ and $M > 0$,

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < M \qquad \text{for } z \in \left\{ z_0 + \epsilon e^{it} : t \in [0,2\pi] \right\}.$$

We can rewrite this as,

$$\left| \int_{\gamma_2} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq 2M\pi\epsilon$$

(1)
$$\left| \int_{\gamma_2} \frac{f(z)}{z - z_0} dz - \int_{\gamma_2} \frac{f(z_0)}{z - z_0} dz \right| \leq 2M\pi\epsilon.$$

Since

$$\int_{\gamma_2} \frac{f(z_0)}{z - z_0} dz = f(z_0) \int_{\gamma_2} \frac{dz}{z - z_0} = 2\pi i f(z_0)$$

(1) can be written as,

$$\left| \int_\gamma \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| \leq 2M\pi\epsilon$$

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz - f(z_0) \right| \leq M\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $f(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(z)}{z - z_0} dz.$ □

DEFINITION 1. We say that a function $f : \Omega \longrightarrow \mathbb{C}$ is Complex analytic if given $z_0 \in \Omega$, there exists a disc $D(z_0, r) \subseteq \Omega$ such that the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $D(z_0, r)$ to the function f.

THEOREM 2. *Let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic on Ω and $z_0 \in \Omega$. Suppose $\overline{D(z_0, r)} \subseteq \Omega$, then for every $n \in \mathbb{N}$, let*

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

where $\gamma(t) = z_0 + re^{it}$ for $t \in [0,2\pi]$. Then the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $D(z_0, r)$ to $f(z)$.

PROOF. Since f is continuous on γ, by compactness, $\exists M > 0$ such that $|f(z)| \leq M$ on γ.

$$|a_n| = \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leq \frac{1}{2\pi}\frac{M}{r^{n+1}}2\pi r = M\frac{1}{r^n}.$$

Hence

$$\limsup |a_n|^{1/n} = \limsup \left(M\frac{1}{r^n}\right)^{1/n} = \frac{1}{r}.$$

Hence $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges in $D(z_0, r)$.

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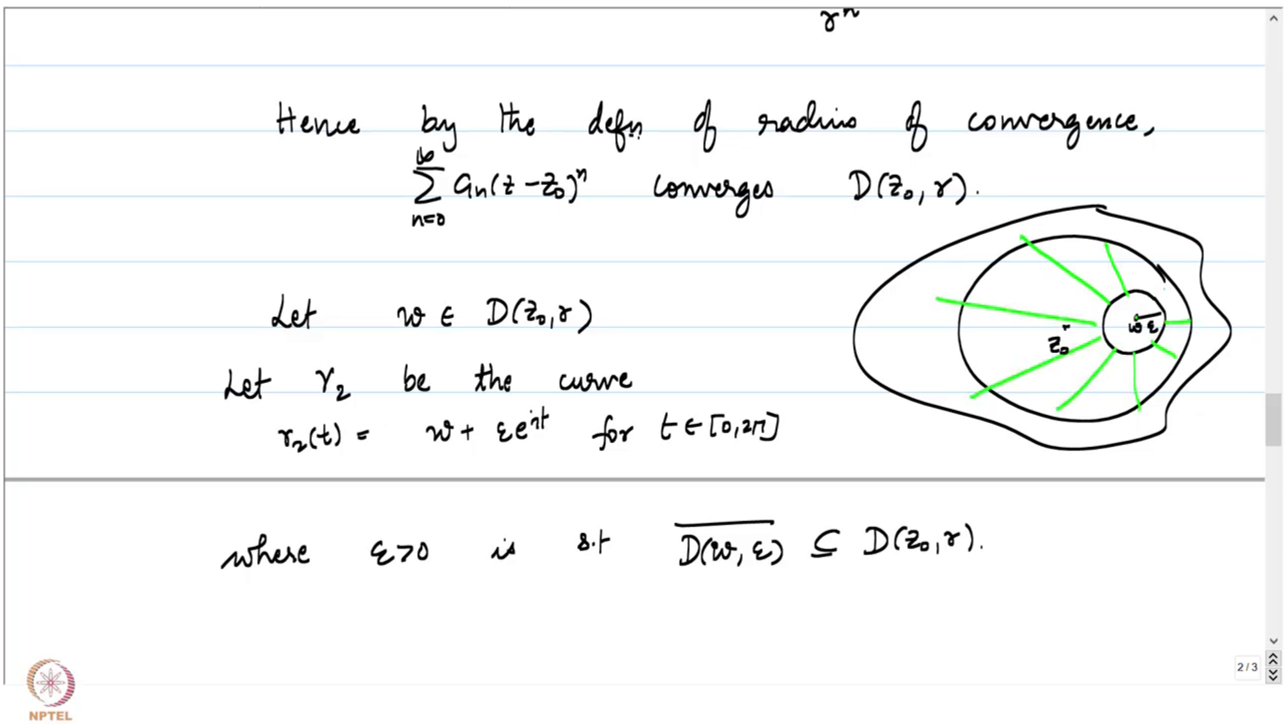

Let $w \in D(z_0, r)$ and γ_2 be the curve $\gamma_2(t) = w + \epsilon e^{it}$ for $t \in [0, 2\pi]$ where $\epsilon > 0$ be such that $\overline{D(w,\epsilon)} \subseteq D(z_0, r)$. In $\overline{D(z_0, r)} \setminus \{w\}$, we have γ_2 is homotopic to γ and hence by the Cauchy integral formula,

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz. \tag{2}$$

We have,

$$\frac{1}{z-w} = \frac{1}{(z-z_0)\left(1-\dfrac{(w-z_0)}{(z-z_0)}\right)}$$

On γ, we have $\left|\dfrac{(w-z_0)}{(z-z_0)}\right| < 1$. Hence,

$$\frac{1}{z-w} = \frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n \tag{3}$$

Substituting (3) in (2), we have

$$f(w) = \frac{1}{2\pi i}\int_{\gamma}\left(f(z)\sum_{n=0}^{\infty}\frac{(w-z_0)^n}{(z-z_0)^{n+1}}\right)dz.$$

Now,

$$\begin{aligned}\left|f(z)\frac{(w-z_0)^n}{(z-z_0)^{n+1}}\right| &\le \frac{M}{r^{n+1}}|w-z_0|^n \\ &\le \frac{M}{r^{n+1}}\rho^n \qquad\qquad \text{where } 0<\rho<1.\end{aligned}$$

By Weierstrass $M-$test, the series $\displaystyle\sum_{n=0}^{\infty} f(z)\frac{(w-z_0)^n}{(z-z_0)^{n+1}}$ converges uniformly and hence we have,

$$f(w) = \frac{1}{2\pi i}\int_{\gamma}\left(\sum_{n=0}^{\infty} f(z)\frac{(w-z_0)^n}{(z-z_0)^{n+1}}\right)dz = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-z_0)^{n+1}}dz\right)(w-z_0)^n.$$

Hence we have $f(w) = \displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ in $D(z_0, r)$. □

COROLLARY 3. *If $f:\Omega \longrightarrow \mathbb{C}$ is holomorphic, then $f':\Omega \longrightarrow \mathbb{C}$ is holomorphic.*

COROLLARY 4. *If $f:\Omega \longrightarrow \mathbb{C}$ is holomorphic, then f is infinitely differentiable.*