Complex Analysis

Prof. Pranav Haridas

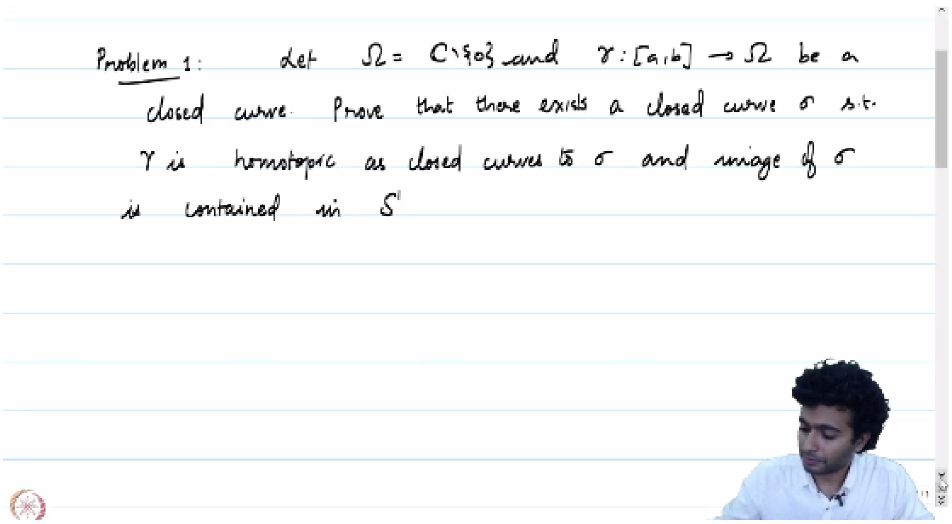
Kerala School of Mathematics

Lecture No – 24

Problem Session

PROBLEM 1. Let $\Omega = \mathbb{C} \setminus \{0\}$ and $\gamma : [a,b] \longrightarrow \Omega$ be a closed curve. Prove that there exists a closed curve σ such that γ is homotopy as closed curves to σ and image of σ is contained in S^1.

(Refer Slide Time: 00:30)



SOLUTION 1. Given $\gamma : [a,b] \longrightarrow \Omega$ is closed curve. That is $\gamma(a) = \gamma(b)$. Define

$$\sigma(t) := \frac{\gamma(t)}{|\gamma(t)|}.$$

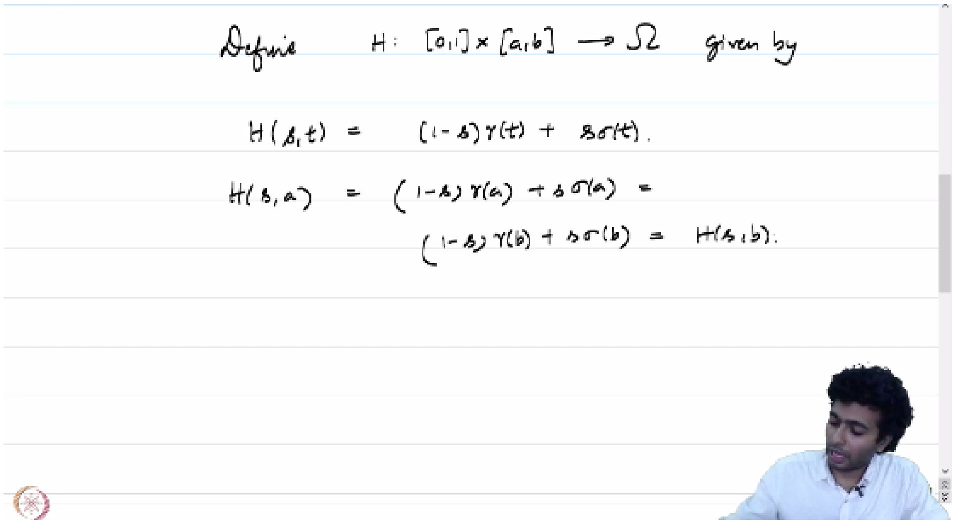
Then

$$\sigma(a) = \frac{\gamma(a)}{|\gamma(a)|} = \frac{\gamma(b)}{|\gamma(b)|} = \sigma(b).$$

Hence we have a closed curve whose image is contained in S^1, which is σ.

It remains to prove that γ is homotopy as closed curves to σ.

(Refer Slide Time: 03:43)



Define

$$H : [0,1] \times [a,b] \longrightarrow \Omega$$

given by

$$H(s,t) = (1-s)\gamma(t) + s\sigma(t)$$

Now it is left as an exercise to the reader to verify that H is indeed a homotopy from γ to σ as closed curve.

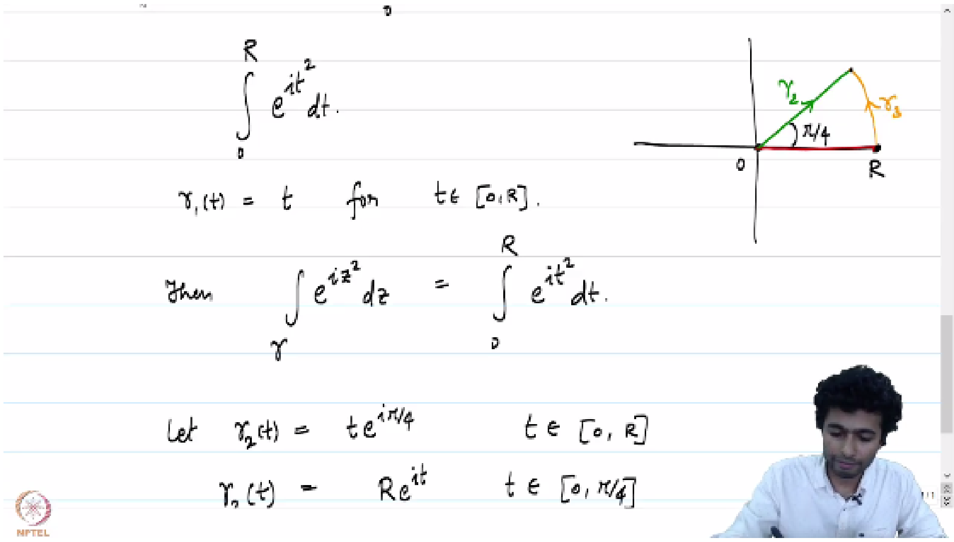
PROBLEM 2. Compute the integral:

$$\int_0^\infty \cos(t^2)\,dt.$$

SOLUTION 2. Notice that $\cos(t^2) = \Re\mathfrak{e}\left(e^{it^2}\right)$ and

$$\int_0^\infty = \lim_{R\to\infty}\int_0^R \cos(t^2)\,dt.$$

(Refer Slide Time: 06:58)



Now, let $\gamma_1(t) = t$ for $t \in [0, R]$. Then, by the change of variable,

$$\int_{\gamma_1} e^{iz^2} dz = \int_0^R e^{it^2} dt.$$

Now lets define another path from 0 to R. Let

$$\gamma_2(t) = te^{i\pi/4} \qquad\qquad t \in [0, R]$$

$$\gamma_3(t) = Re^{it} \qquad\qquad t \in [0, \pi/4].$$

Then $\gamma_2(t) + (-\gamma_3)$ is a curve from 0 to R.

Since e^{iz^2} is an entire function, by Cauchy's theorem,

$$\int_{\gamma_1} e^{iz^2} dz = \int_{\gamma_2 + (-\gamma_3)} e^{iz^2} dz.$$

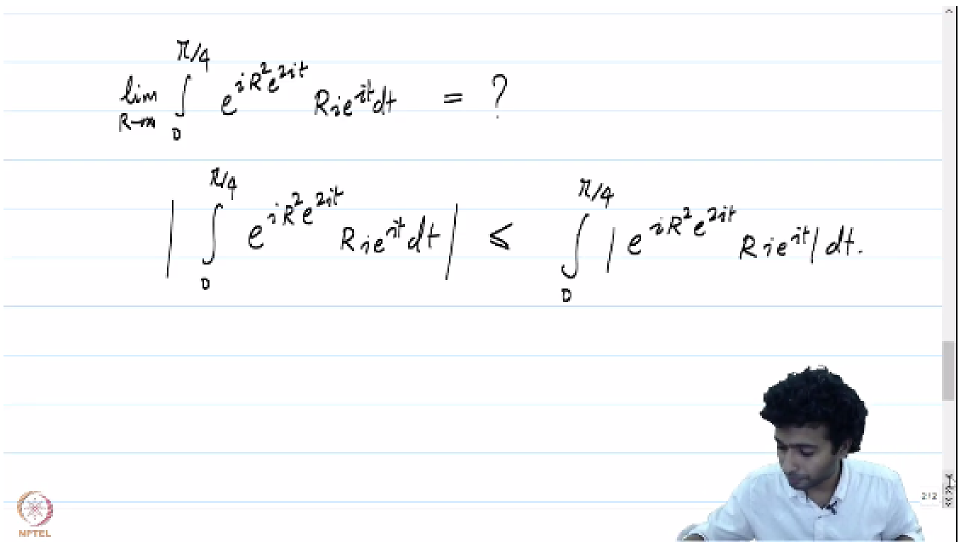
$$\begin{aligned}
\int_{\gamma_2 + (-\gamma_3)} e^{iz^2} dz &= \int_{\gamma_2} e^{iz^2} dz - \int_{\gamma_3} e^{iz^2} dz \\
&= \int_0^R e^{i(te^{i\pi/4})^2} e^{i\pi/4} dt - \int_0^{\pi/4} e^{i(Re^{it})^2} Rie^{it} dt. \\
&= \int_0^R e^{-t^2} \left(\frac{1+i}{\sqrt{2}}\right) dt - \int_0^{\pi/4} e^{iR^2 e^{2it}} Rie^{it} dt.
\end{aligned}$$

Now,

$$\lim_{R\to\infty} \frac{(1+i)}{\sqrt{2}} \int_0^R e^{-t^2}\, dt = \frac{1+i}{\sqrt{2}} \int_0^\infty e^{-t^2}\, dt$$
$$= \frac{1+i}{\sqrt{2}} \left(\frac{\sqrt{\pi}}{2}\right)$$

Now let's consider the term $\int_0^{\pi/4} e^{iR^2 e^{2it}} Rie^{it}\, dt.$

(Refer Slide Time: 15:30)



We know that $e^{2it} = \cos(2t) + i\sin(2t)$. Now,

$$\left|\int_0^{\pi/4} e^{iR^2 e^{2it}} Rie^{it}\, dt\right| \le \int_0^{\pi/4} \left|e^{iR^2 e^{2it}} Rie^{it}\right| dt$$
$$\le R\int_0^{\pi/4} e^{-R^2 \sin(2t)}\, dt$$
$$\le R\int_0^{\pi/4} e^{-R^2 t}\, dt$$
$$= -\frac{R}{R^2}\left(e^{-R^2 t}\right)\Big|_0^{\pi/4}$$
$$= \frac{1}{R}\left(1 - e^{-R^2\pi/4}\right)$$
$$\le \frac{1}{R}.$$

Hence,

$$\lim_{R\to\infty}\int_{\gamma_3} e^{z^2}\,dz = 0.$$

Hence

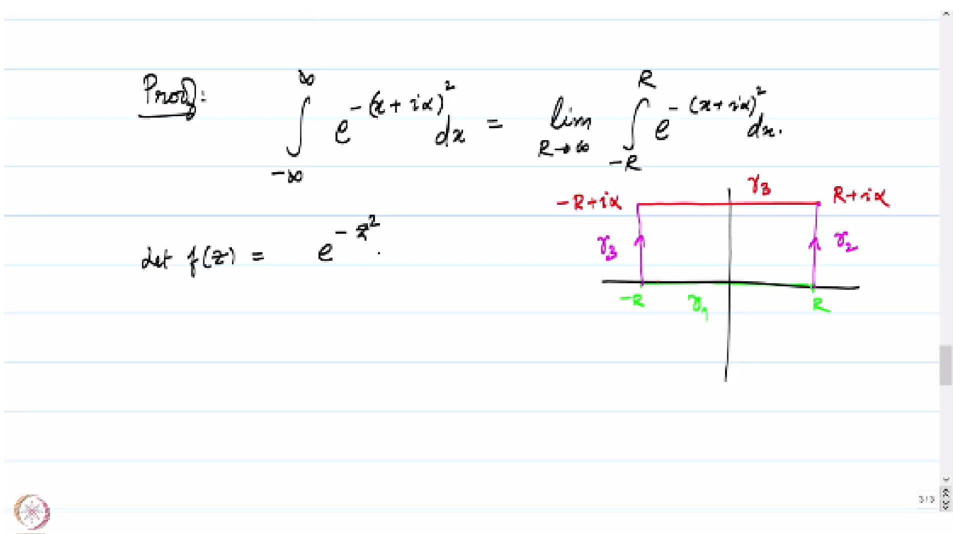
$$\int_0^\infty = \frac{1+i}{\sqrt{2}}\frac{\sqrt{\pi}}{2}.$$

$$\int_0^\infty \cos(t^2)\,dt = \Re\mathfrak{e}\int_0^\infty e^{it^2}\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}} \qquad ; \quad \int_0^\infty \sin(t^2)\,dt = \Im\mathfrak{m}\int_0^\infty e^{it^2}\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

PROBLEM 3. Prove that

$$\int_{-\infty}^{\infty} e^{-(x+i\alpha)^2}\,dx = \int_{\infty}^{\infty} e^{-x^2}\,dx$$

SOLUTION 3. **(Refer Slide Time: 21:33)**

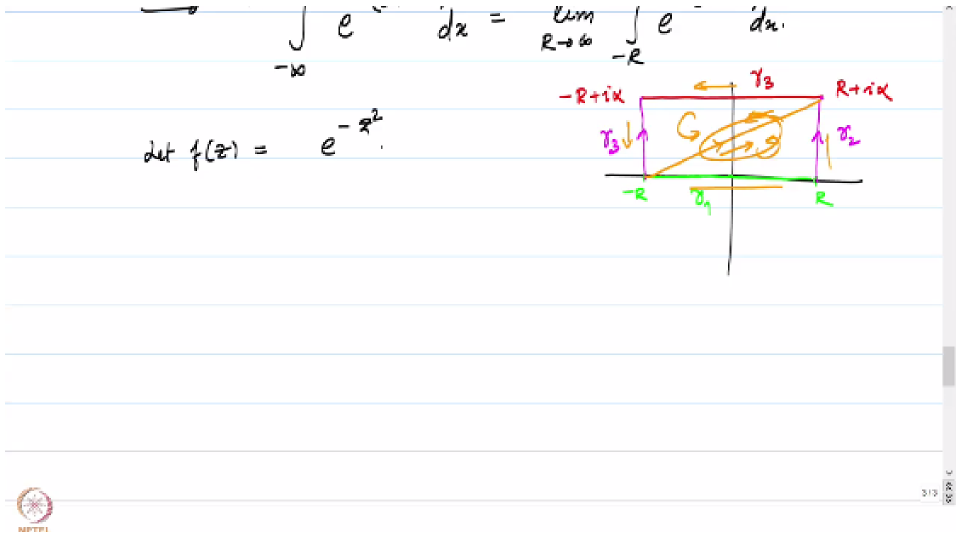


We can rewrite the L.H.S in the problem as,

$$\int_{-\infty}^{\infty} e^{-(x+i\alpha)^2}\,dx = \lim_{R\to\infty}\int_{-R}^{R} e^{-(x+i\alpha)^2}\,dx.$$

Let $f(z) = e^{-z^2}$. Then f is an entire function.

(Refer Slide Time: 23:37)



Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be straight lines such that,

$$\gamma_1 : -R \longrightarrow R$$

$$\gamma_2 : R \longrightarrow R + i\alpha$$

$$\gamma_3 : R + i\alpha \longrightarrow -R + i\alpha$$

$$\gamma_4 : -R \longrightarrow -R + i\alpha.$$

Then we can parametrize these curves as follows:

$$\gamma_1(t) = t \qquad t \in [-R, R]$$

$$\gamma_2(t) = R + it \qquad t \in [0, \alpha]$$

$$\gamma_3(t) = -t + i\alpha \qquad t \in [-R, R]$$

$$\gamma_4(t) = -R + it \qquad t \in [0, \alpha].$$

Let us compute the integral of f along all these curves,

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{R} e^{-t^2} dt.$$

$$\int_{\gamma_2} f(z)dz = \int_0^\alpha e^{-((R^2-t^2)+2iRt)}\, i\, dt$$

$$\Longrightarrow \left|\int_{\gamma_2} f(z)dz\right| = \left|\int_0^\alpha e^{-((R^2-t^2)+2iRt)}\, i\, dt\right|$$

$$\leq \int_0^\alpha e^{-(R^2-t^2)}dt$$

$$\leq \int_0^\alpha e^{(R^2-\alpha^2)}dt$$

$$= \frac{\alpha e^{\alpha^2}}{e^{R^2}} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty.$$

Let us consider the integral of f over $-\gamma_3$,

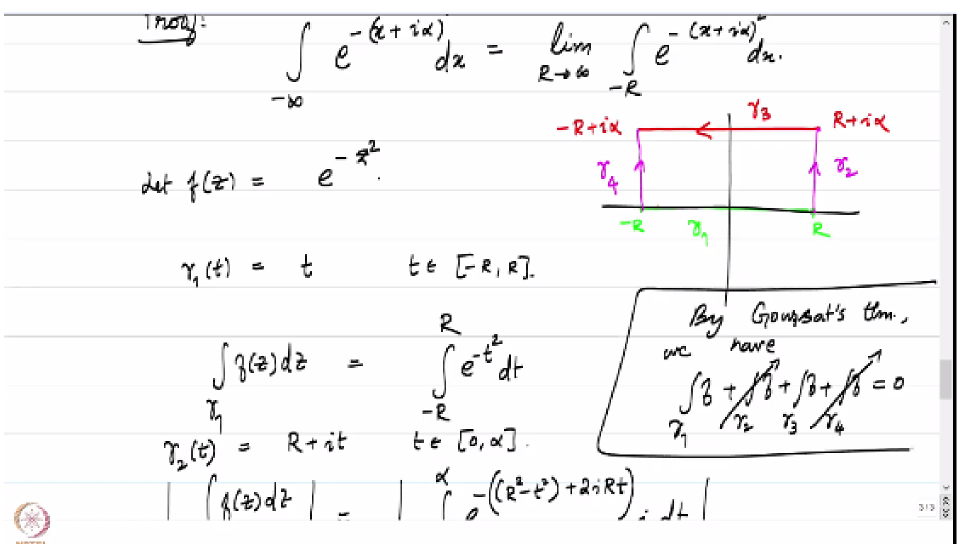
$$\int_{(-\gamma_3)} f(z)dz = \int_{-R}^{R} e^{-(t+i\alpha)^2}dt.$$

$$\lim_{R\to\infty}\int_{(-\gamma_3)} f(z)dz = \int_{-\infty}^{\infty} e^{-(t+i\alpha)^2}dt.$$

Since the case of integral over γ_4 is also going to similar as in the case of γ_2, we have

$$\int_{\gamma_4} f(z)dz \longrightarrow 0 \quad \text{as} \quad R \longrightarrow \infty.$$

(Refer Slide Time: 31:19)



By Cauchy-Goursat theorem,

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{(-\gamma_4)} f(z)dz = 0$$

$$\implies \int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz = 0 \qquad \text{as } R \longrightarrow \infty$$

$$\int_{\gamma_1} f(z)dz = \int_{(-\gamma_3)} f(z)dz$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-(x+i\alpha)^2} dx.$$

PROBLEM 4. Let Ω be a simply connected domain and $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function such that $f(z) \neq 0 \; \forall z \in \Omega$ and f' is also holomorphic. Then there exists $g : \Omega \longrightarrow \mathbb{C}$ such that

$$f(z) = e^{g(z)}.$$

SOLUTION 4. Consider the function $\dfrac{f'(z)}{f(z)}$. Since f is non-vanishing on Ω, $\dfrac{f'(z)}{f(z)}$ is holomorphic on Ω. Since Ω is simply connected, there exists an anti-derivative $g_0(z)$ such that $g_0'(z) = \dfrac{f'(z)}{f(z)}$. Define $h(z) = e^{(g_0(z))}$.

Consider

$$\left(\frac{f(z)}{h(z)}\right)' = \frac{f'(z)h(z) - f(z)h'(z)}{(h(z))^2}.$$

Now,

$$f'(z)h(z) - f(z)h'(z) = f'(z)e^{g_0(z)} - f(z)e^{g_0(z)}\frac{f'(z)}{f(z)} = f'(z)e^{g_0(z)} - e^{g_0(z)}f'(z) = 0.$$

Hence, on Ω,

$$\left(\frac{f(z)}{h(z)}\right)' = 0$$

$$f(z) = ch(z) = ce^{g_0(z)}$$

Let $c' \in \mathbb{C}$ be such that $e^{c'} = c$, then $f(z) = e^{c' + g_0(z)}$. Put $g(z) = c' + g_0(z)$, then g is holomorphic on Ω and $f = e^g$.