Complex Analysis

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Lecture No – 23

Cauchy’s Theorem

In the last lecture as a corollary to Goursat theorem, we proved that if Ω is a convex domain and if γ is a closed polygonal path in Ω, then for any holomorphic function f on Ω, $\int_\gamma f(z)dz = 0$. By using a complex analog of the second fundamental theorem of calculus, which was proved in the last week, we can immediately conclude a version of Cauchy’s theorem in the case of convex domains. This is sometimes also called the local version of the Cauchy’s theorem.

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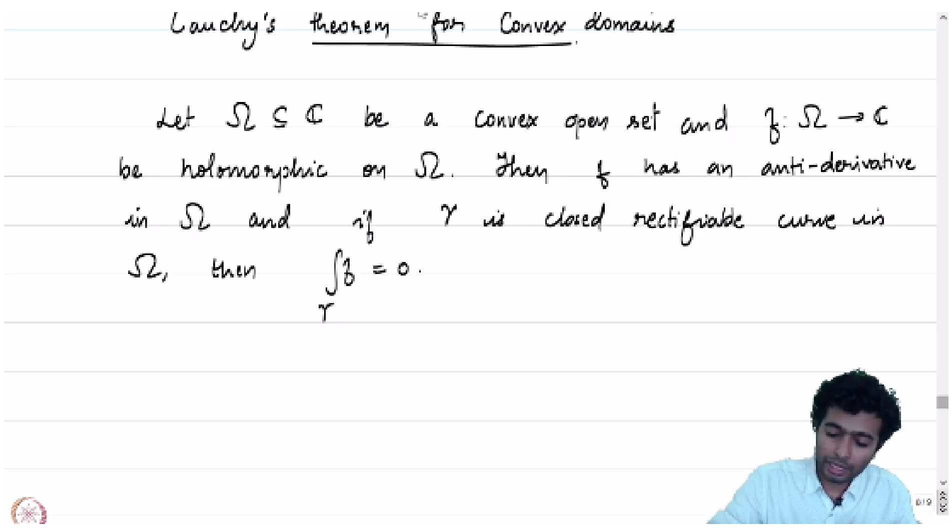

THEOREM 1 (**Cauchy’s Theorem for Convex Domains**).
Let $\Omega \subseteq \mathbb{C}$ be a convex open set and $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic on Ω. Then f has an anti-derivative in Ω and if γ is closed rectifiable curve in Ω, then

$$\int_\gamma f(z)dz = 0.$$

PROOF. By Cauchy's theorem of polygonal paths, we have $\int_\sigma f(z)dz = 0$ for any closed polygonal path σ in Ω. Then by the second fundamental theorem of calculus, we have an explicit anti-derivative F of f on Ω which was given by

$$F(z_1) = \int_\gamma f(z)dz$$

where γ is the straight line from z_0 to z_1 for $z_0 \in \Omega$ fixed.

By the first fundamental theorem of calculus,

$$\int_\gamma f(z)dz = 0$$

for every rectifiable curve γ in Ω since f has an anti-derivative. □

THEOREM 2 (**Cauchy's Theorem**).
Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic on Ω. Let $\gamma_0 : [a,b] \longrightarrow \Omega$ and $\gamma_1 : [c,d] \longrightarrow \Omega$ be rectifiable curves on Ω from z_0 to z_1 and such that γ_0 is homotopic with fixed end-points to a reparametrization of γ_1, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

PROOF. Since the integral is invariant under reparametrization, we may assume that γ_1 is defined on $[a,b]$.

Let $H : [0,1] \times [a,b] \longrightarrow \Omega$ be a homotopy with fixed end-points from γ_0 to γ_1. Since $[0,1] \times [a,b]$ is compact, we have $H([0,1] \times [a,b])$ is compact in Ω.

Let $\mathscr{U} = \{U_1, U_2, \ldots, U_n\}$ be open subsets of Ω such that $\overline{U}_j \subseteq \Omega$ and $H([0,1] \times [a,b]) \subset \bigcup_{j=1}^{n} U_j$. Let r be the Lebesgue number corresponding to $\mathscr{U}$. Then for any $(s,t) \in [0,1] \times [a,b]$, $D(H(s,t), r) \subseteq \Omega$.

Since $[0,1] \times [a,b]$ is compact, H is uniformly continuous on $[0,1] \times [a,b]$ and hence $\exists \delta > 0$ such that

$$|H(s,t) - H(s',t')| < \frac{r}{4} \quad \text{whenever } |s - s'| < \delta \text{ and } |t - t'| < \delta.$$

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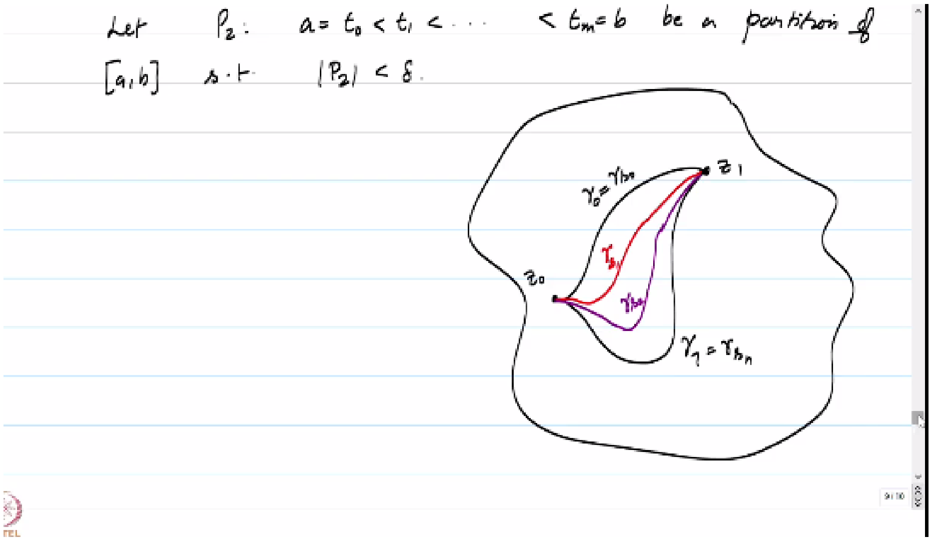

Let $P_1 : 0 = s_0 < s_1 < \cdots < s_n = 1$ be a partition of $[0,1]$ such that $|P_1| < \delta$ and let $P_2 : a = t_0 < t_1 < \cdots < t_m = b$ be a partition of $[a,b]$ such that $|P_2| < \delta$.

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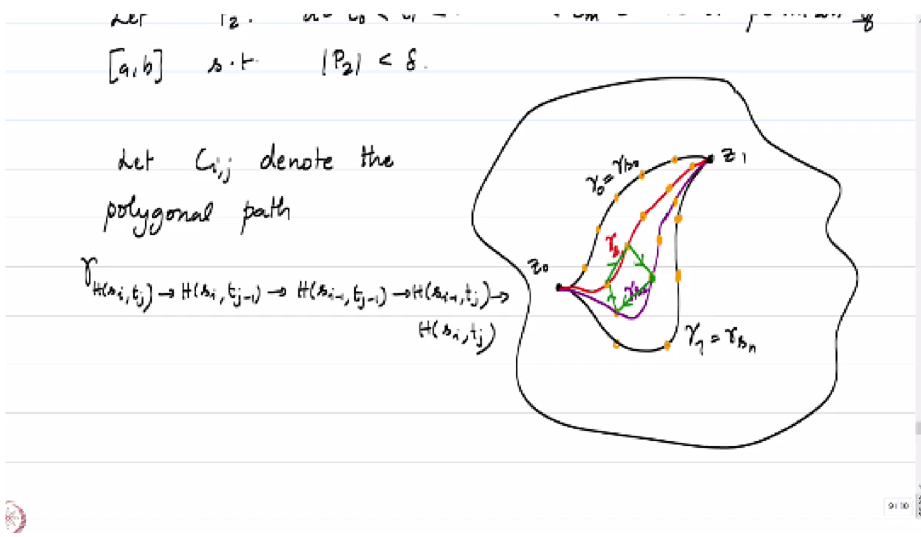

Let $C_{i,j}$ denote the polygonal path $\gamma_{H(s_i,t_j) \to H(s_i,t_{j-1}) \to H(s_{i-1},t_{j-1}) \to H(s_{i-1},t_j) \to H(s_i,t_j)}$. By our choice of δ, we can ensure that $\text{diam}(C_{i,j}) < r$. Hence $C_{i,j} \subset D(H(s_i,t_j), r)$.

By Cauchy's theorem for polygonal paths on convex open sets, we have

$$\int_{C_{i,j}} f(z)dz = 0.$$

Now it is left as an exercise to the reader to verify that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{C_{i,j}} f(z)dz = \int_{C} f(z)dz$$

where $C := \gamma_{H(1,t_m)\to H(1,t_{m-1})\to\cdots\to H(1,t_0)\to H(0,t_1)\to\cdots\to H(0,t_m)}$. Then C is a reparametrization of $(-\sigma_1)+\sigma_2$, where

$$\sigma_1 := \gamma_{H(1,t_0)\to\cdots\to H(1,t_m)}$$

$$\sigma_2 := \gamma_{H(0,t_0)\to\cdots\to H(0,t_m)}.$$

Then

$$\int_{C} f(z)dz = \int_{(-\sigma_1)+\sigma_2} f(z)dz = \int_{\sigma_2} f(z)dz - \int_{\sigma_1} f(z)dz.$$

Since

$$\int_{C_{i,j}} f(z)dz = 0 \qquad \forall\ i,j,$$

we have

$$\int_{C} f(z)dz = 0 \implies \int_{\sigma_1} f(z)dz = \int_{\sigma_2} f(z)dz.$$

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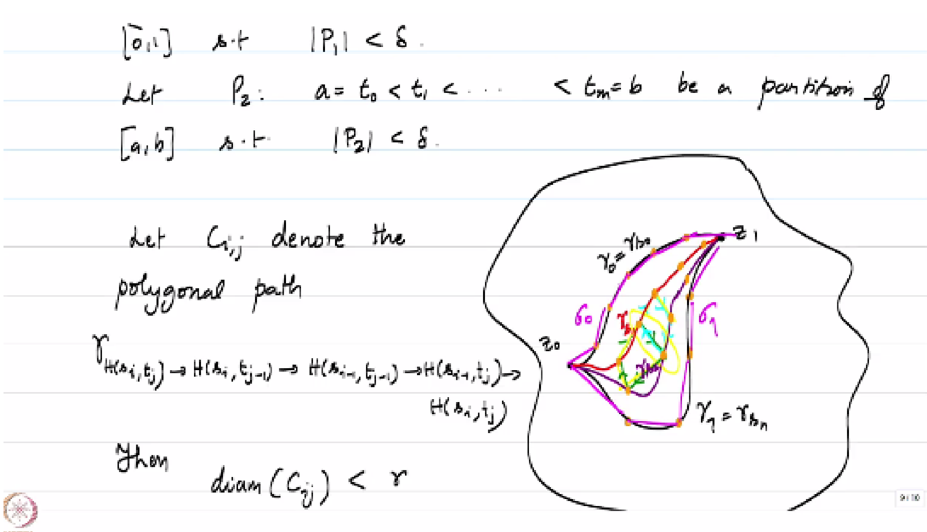

Notice that $\gamma_0 \restriction_{[t_j, t_{j+1}]} ([t_j, t_{j+1}]) \subseteq D(H(0,t_j), r)$ by our choice of partition. Similarly, $\gamma_{H(0,t_j) \to H(0,t_{j+1})}([0,1]) \subseteq D(H(0,t_j), r)$.

By Cauchy's theorem on convex sets,

$$\int_{\gamma_0 \restriction_{[t_j, t_{j+1}]}} f(z)dz = \int_{\gamma_{H(0,t_j) \to H(0,t_{j+1})}} f(z)dz.$$

$$\begin{aligned}
\int_{\sigma_2} f(z)dz &= \int_{\gamma_{H(0,t_0) \to \cdots \to H(0,t_m)}} f(z)dz \\
&= \sum_{j=0}^{m-1} \int_{\gamma_{H(0,t_j) \to H(0,t_{j+1})}} f(z)dz \\
&= \sum_{j=0}^{m-1} \int_{\gamma_0 \restriction_{[t_j, t_{j+1}]}} f(z)dz \\
&= \int_{\gamma_0 \restriction_{[0,t_1]} + \cdots + \gamma_{[t_{m-1}, t_m]}} f(z)dz \\
&= \int_{\gamma_0} f(z)dz.
\end{aligned}$$

Similarly we can establish that

$$\int_{\sigma_1} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Hence we can conclude that

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

□

DEFINITION 1 (**Simply Connected Domains**). An open set $\Omega \subseteq \mathbb{C}$ is said to be simply connected if every closed curve γ at $z_0 \in \Omega$ is null homotopic to the constant curve at z_0.

Now we can state a special case of Cauchy's theorem for simply connected domains.

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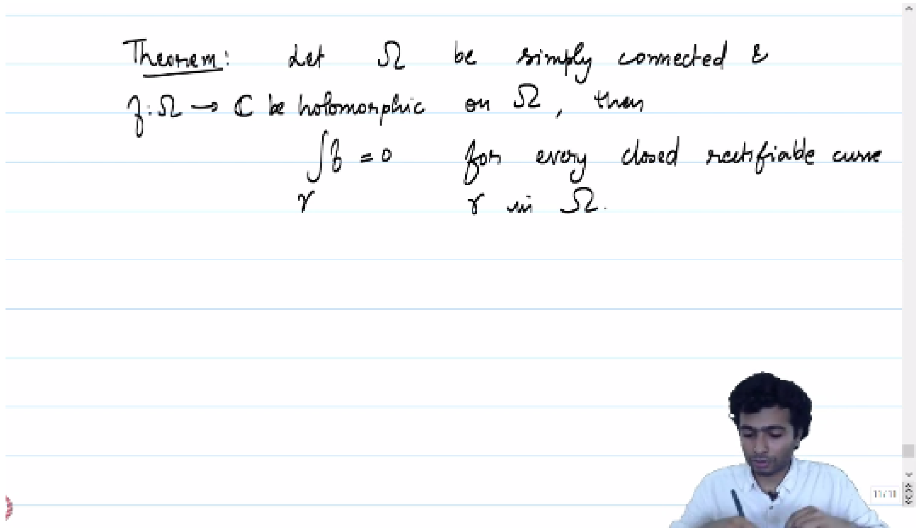

THEOREM 3. *Let $\Omega \subseteq \mathbb{C}$ be simply connected and $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic on Ω, then*

$$\int_\gamma f(z)dz = 0$$

for every closed rectifiable curve γ in Ω.

Proof of the theorem is immediate from the Cauchy's theorem and the definition of simply connected domain.