Complex Analysis

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Lecture No – 22

Cauchy-Goursat Theorem

In the last lecture we laid the topological framework to state Cauchy's theorem.

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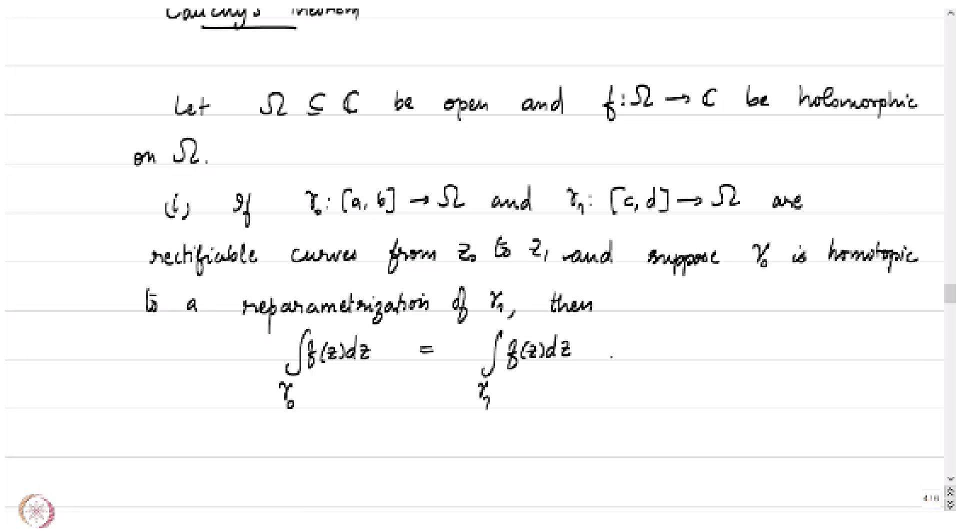

Cauchy's Theorem

Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic on Ω.

(i) If $\gamma_0 : [a, b] \longrightarrow \Omega$ and $\gamma_1 : [c, d] \longrightarrow \Omega$ are rectifiable curves from z_0 to z_1 and suppose γ_0 is homotopic to a reparametrization of γ_1, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

(ii) If $\gamma_0 : [a, b] \longrightarrow \Omega$ and $\gamma_1 : [c, d] \longrightarrow \Omega$ are closed rectifiable curves such that γ_0 is homotopic as closed curves to a reparametrization of γ_1, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Cauchy's theorem is a very powerful tool in the sense that it can be used to compute the integral of f over any rectifiable curve by shifting the curve via homotopy to a contour over which the integral can be computed easily.

We will now state variant of the Cauchy's theorem. It essentially says the same.

Cauchy's Theorem-II

Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic on Ω. Suppose $\gamma_0 : [a,b] \longrightarrow \Omega$ is a rectifiable curve which is null-homotopic. Then

$$\int_{\gamma_0} f(z)dz = 0.$$

Let us compare the two variance of Cauchy's theorem.

Cauchy's theorem $\implies$ Cauchy's theorem-II is immediate, because in the part (ii) of Cauchy's theorem, if we take γ_1 to be a constant curve, then the hypothesis of Cauchy's theorem-II is satisfied. Conclusion follows from the fact that $\int_{\gamma_1} f(z)dz = 0$, where γ_1 is a constant curve.

Proving the other implication is non-trivial. We may put that as a proposition.

PROPOSITION 1. *Cauchy's theorem-II $\implies$ Cauchy's theorem.*

PROOF. Here we will prove part (i) of Cauchy's theorem and leave part (ii) as an exercise to the reader.

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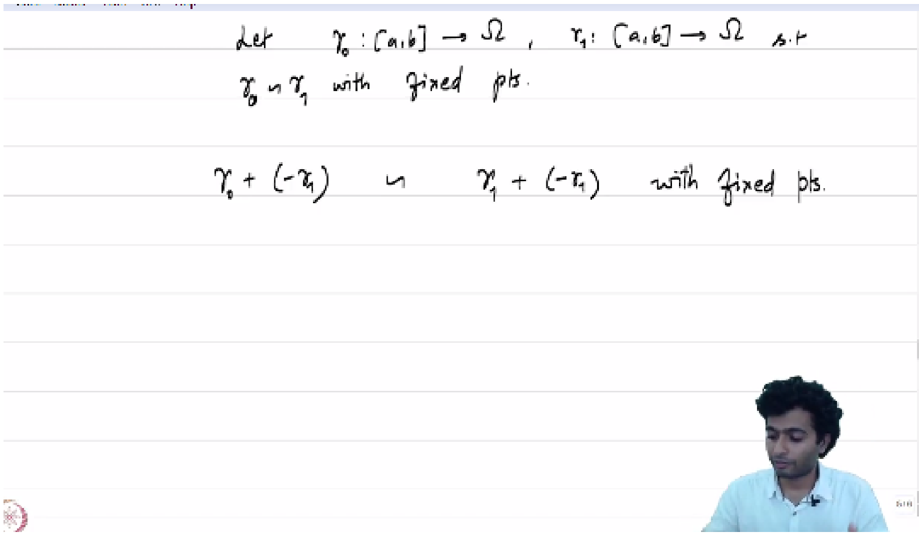

Let $\gamma_0 : [a,b] \longrightarrow \Omega, \gamma_1 : [a,b] \longrightarrow \Omega$ such that $\gamma_0 \sim \gamma_1$ with fixed end points.

$$\gamma_0 + (-\gamma_1) \sim \gamma_1 + (-\gamma_1) \quad \text{with fixed end points}$$

$$\sim \gamma_2 \quad \text{with fixed end points}$$

where $\gamma_2 : [a,b] \longrightarrow \Omega$ is such that $\gamma_2(t) = z_0 = \gamma_0(a)$.

By Cauchy's theorem-II, we have

$$\int_{\gamma_0 + (-\gamma_1)} f(z)dz = 0 \iff \int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

$\square$

EXAMPLE 2. Let $f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$ given by $f(z) = \dfrac{1}{z}$. Consider $\gamma(t) = e^{it}, \ t \in [0, 2\pi]$. Then

$$\int_{\gamma} f(z)dz = 2\pi i.$$

THEOREM 3 (**Goursat's Theorem**).

Let $\Omega \subseteq \mathbb{C}, f : \Omega \longrightarrow \mathbb{C}$ *be holomorphic and* $z_1, z_2, z_3 \in \Omega$ *such that the convex hull of* z_1, z_2, z_3 *is contained in* Ω*. Then*

$$\int_{\gamma_{z_1 \to z_2 \to z_3 \to z_1}} f(z)dz = 0.$$

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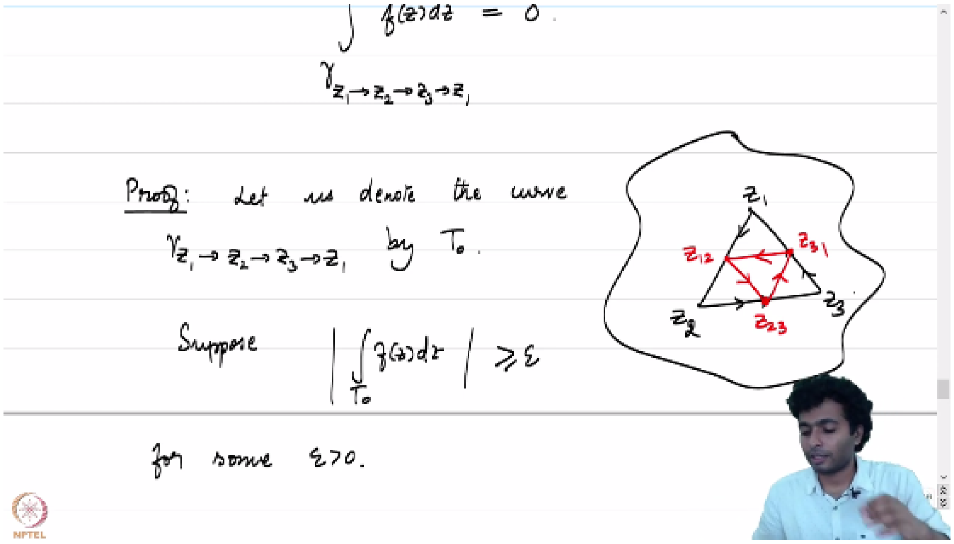

PROOF. Let us denote the curve $\gamma_{z_1\to z_2\to z_3\to z_1}$ by T_0.

Suppose

$$\left|\int_{T_0} f(z)dz\right| \geq \epsilon$$

for some $\epsilon > 0$. Let z_{12} be the midpoint of the straight line joining z_1 and z_2. Similarly z_{23}, z_{31} be the midpoints of straight-line joining z_2 and z_3, and z_3 and z_1 respectively.

Then

$$\left|\int_{T_0} f(z)dz\right| = \left|\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz\right|,$$

where,

$\gamma_1 = \gamma_{z_1\to z_{12}\to z_{31}\to z_1}, \gamma_2 = \gamma_{z_2\to z_{23}\to z_{12}\to z_2}, \gamma_3 = \gamma_{z_3\to z_{31}\to z_{23}\to z_3}$ and $\gamma_4 = \gamma_{z_{12}\to z_{23}\to z_{31}\to z_{12}}$.

Thus,

$$\epsilon \leq \left|\int_{T_0} f(z)dz\right| \leq \left|\int_{\gamma_1} f(z)dz\right| + \left|\int_{\gamma_2} f(z)dz\right| + \left|\int_{\gamma_3} f(z)dz\right| + \left|\int_{\gamma_4} f(z)dz\right|.$$

Hence at least one of $\gamma_1, \gamma_2, \gamma_3$ and γ_4, which shall be denoted as T_1, must satisfy

$$\left|\int_{T_1} f(z)dz\right| \geq \frac{\epsilon}{4}.$$

Note that $|T_1| = \dfrac{|T_0|}{2}$.

Let $\hat{T}_i$ denote the convex hull of T_i and let $\operatorname{diam}(\hat{T}_i) = \sup\left\{|z-w| : z, w \in \hat{T}_i\right\}$.

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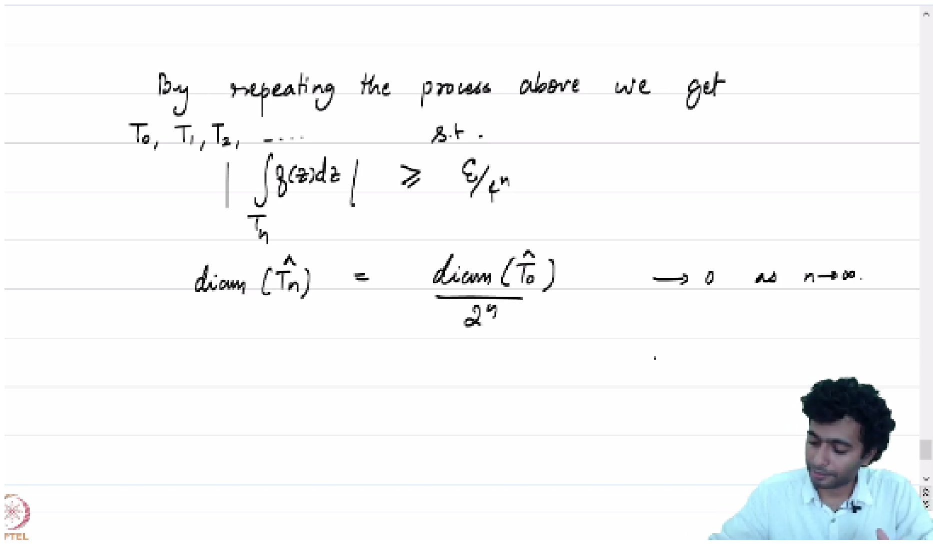

Now, $\operatorname{diam}(\hat{T}_1) = \dfrac{\operatorname{diam}(\hat{T}_0)}{2}$. By repeating the process above, we get $T_0, T_1, T_2, \cdots$ such that

$$\left|\int_{T_n} f(z)dz\right| \geq \frac{\epsilon}{4^n} \longrightarrow (*).$$

Also observe that $\hat{T}_0 \supset \hat{T}_1 \supset \cdots$.

$$|T_n| = \frac{|T_0|}{2^n},\ \operatorname{diam}(\hat{T}_n) = \frac{\operatorname{diam}(\hat{T}_0)}{2^n}.$$

Then $|T_n| \to 0$ and $\operatorname{diam}(\hat{T}_n) \to 0$ as $n \to \infty$.

Pick $z_n \in \hat{T}_n$ and then $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence by our choice of $\hat{T}_i$. By completeness, $z_n \to z_0$ where $z_0 \in \bigcap_{i=0}^{\infty} \hat{T}_i$.

Since f is complex differentiable at z_0, given $\epsilon' > 0$, $\exists\, \delta > 0$ such that

$$\left|\left(f(z) - f(z_0)\right) - f'(z_0)(z - z_0)\right| < \epsilon' |z - z_0|$$

whenever $|z - z_0| < \delta$.

For large n, $\hat{T}_n \subset D(z_0, \delta)$. Hence $|z - z_0| < \text{diam}(\hat{T}_n) = \dfrac{\text{diam}(\hat{T}_0)}{2^n}$.

That is,

$$\left|\left(f(z) - f(z_0)\right) - f'(z_0)(z - z_0)\right| < \epsilon' \frac{\text{diam}(\hat{T}_0)}{2^n}.$$

Hence

$$\left|\int_{T_n} \left(f(z) - \left(f(z_0) + f'(z_0)(z - z_0)\right)\right) dz\right| \le \epsilon' |T_n| \frac{\text{diam}(\hat{T}_0)}{2^n} = \epsilon' |T_0| \frac{\text{diam}(\hat{T}_0)}{4^n}.$$

That is,

$$\left|\int_{T_n} f(z) dz - \int_{T_n} \left(f(z_0) + f'(z_0)(z - z_0)\right) dz\right| \le \epsilon' |T_0| \frac{\text{diam}(\hat{T}_0)}{4^n}.$$

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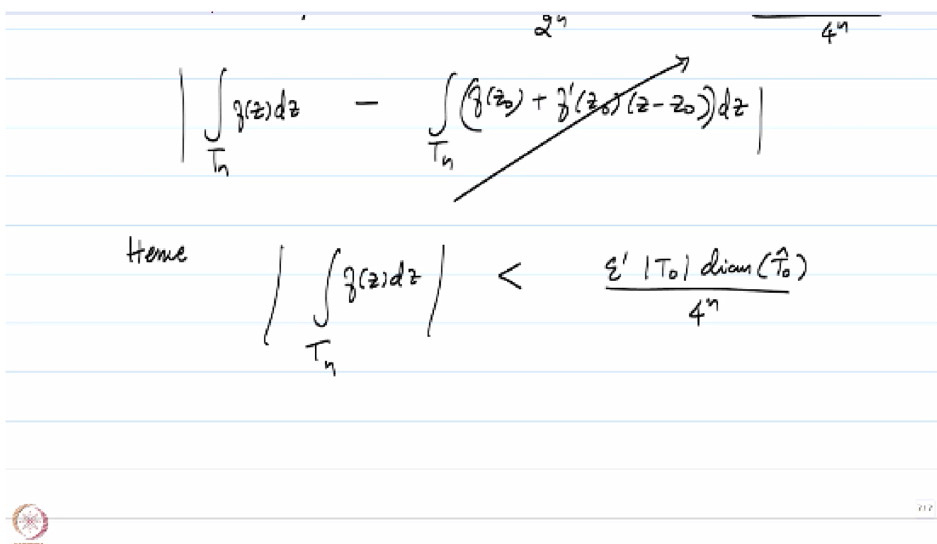

Now it is left to reader to verify by using first fundamental theorem of calculus that,

$$\int_{T_n} \left(f(z_0) + f'(z_0)(z - z_0)\right) dz = 0.$$

Hence

$$\left|\int_{T_n} f(z) dz\right| \le \epsilon' |T_0| \frac{\text{diam}(\hat{T}_0)}{4^n}.$$

Then for ϵ' small enough such that $\epsilon'|T_0|\frac{\text{diam}(\hat{T}_0)}{<}\epsilon$,

$$\left|\int_{T_n} f(z)dz\right| < \frac{\epsilon}{4^n}$$

which is a contradiction to $(*)$.

Thus,

$$\left|\int_{T_0} f(z)dz\right| < \epsilon \quad \forall \epsilon > 0$$

$$\implies \int_{T_0} f(z)dz = 0.$$

$\square$

THEOREM 4 (**Cauchy's Theorem of Polygonal Paths**).
Let $\Omega \subseteq \mathbb{C}$ *be a convex and* $\gamma_{z_1 \to z_2 \to \cdots \to z_n \to z_1}$ *be a closed polygonal path and suppose* $f : \Omega \longrightarrow \mathbb{C}$ *be holomorphic. Then*

$$\int_{\gamma_{z_1 \to z_2 \to \cdots \to z_n \to z_1}} f(z)dz = 0.$$

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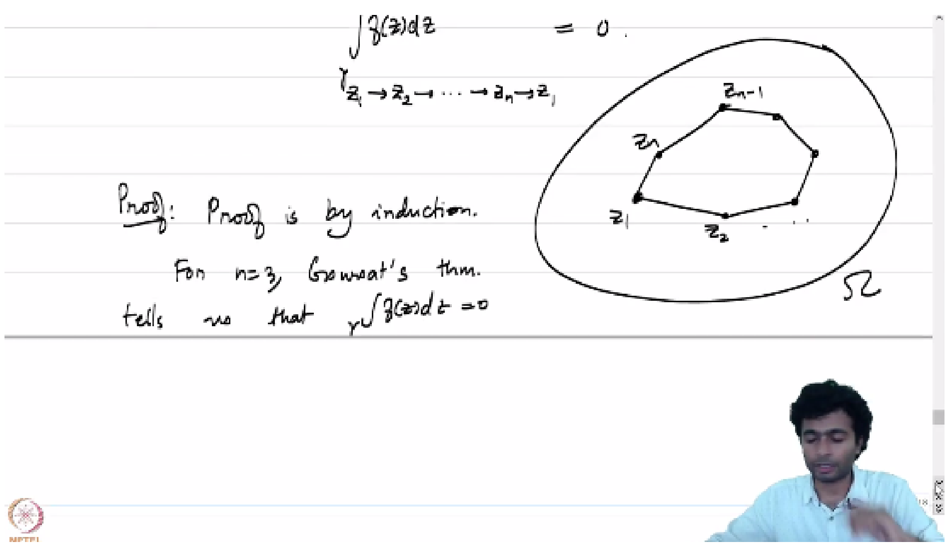

PROOF. Proof is by using induction. For the base case, we take $n = 3$ because for $n = 1, 2$ we have nothing to prove. For $n = 3$ Goursat's theorem tells us that

$$\int_{\gamma} f(z)dz = 0$$

where γ is a closed polygonal path with 3 vertices.

Assume the result is proved for up to $n-1$. Then,

$$\begin{aligned}\int_{\gamma_{z_1 \to z_2 \to \cdots \to z_n \to z_1}} f(z)dz &= \int_{\gamma_{z_1 \to z_2 \to \cdots \to z_{n-1} \to z_1}} f(z)dz + \int_{\gamma_{z_{n-1} \to z_n \to z_1 \to z_{n-1}}} f(z)dz \\ &= 0 + 0 = 0\end{aligned}$$

$\square$