Complex Analysis

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Lecture No – 20

Problem Session

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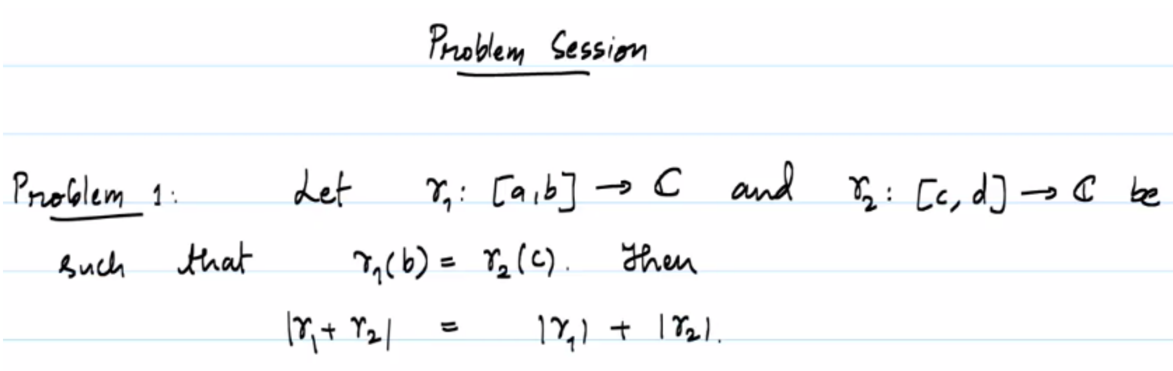


PROBLEM 1. Let $\gamma_1 : [a,b] \longrightarrow \mathbb{C}$ and $\gamma_2 : [c,d] \longrightarrow \mathbb{C}$ be such that $\gamma_1(b) = \gamma_2(c)$. Then

$$|\gamma_1 + \gamma_2| = |\gamma_1| + |\gamma_2|.$$

SOLUTION 1. The first observation is that if you look at a continuous re-parametrization of a given curve, then the re-parameterized curve also will have the same arc length. This was given as an exercise to the reader to verify.

Assuming this exercise, without loss of generality, we may assume that $b = c$.

(Refer Slide Time: 04:12)

$$|\gamma_1+\gamma_2| - \sum_{j=1}^{n} \left|(\gamma_1+\gamma_2)(t_j) - (\gamma_1+\gamma_2)(t_{j-1})\right| < \varepsilon/3$$

and s.t. $|\gamma_1| - \sum_{j=1}^{k} |\gamma_1(t_j) - \gamma_1(t_{j-1})| < \varepsilon/3$

and s.t $|\gamma_2| - \sum_{j=k+1}^{n} |\gamma_2(t_j) - \gamma_2(t_{j-1})| < \varepsilon/3.$

Let $\epsilon > 0$ be given. Let $P : a = t_0 < t_1 < \cdots < b = t_k = c < t_{k+1} < \cdots < t_n = b$ be a partition of $[a, d] = [a, b] \cup [b, d]$ such that

$$|\gamma_1+\gamma_2| - \sum_{j=1}^{n} \left|(\gamma_1+\gamma_2)(t_j) - (\gamma_1+\gamma_2)(t_{j-1})\right| < \frac{\epsilon}{3} \tag{1}$$

and such that

$$|\gamma_1| - \sum_{j=1}^{k} \left|\gamma_1(t_j) - \gamma_1(t_{j-1})\right| < \frac{\epsilon}{3}, \tag{2}$$

$$|\gamma_2| - \sum_{j=k+1}^{n} \left|\gamma_2(t_j) - \gamma_2(t_{j-1})\right| < \frac{\epsilon}{3}. \tag{3}$$

Existence of such a partition can always ensure because there exists a partition Q_1 such that (1) is satisfied, similarly there exists partitions Q_2 and Q_3 of $[a, b]$ and $[b, d]$ respectively such that (2) and (3) are satisfied. Now if we take a common refinement of Q_1, Q_2 and Q_3, then all these conditions are satisfied.

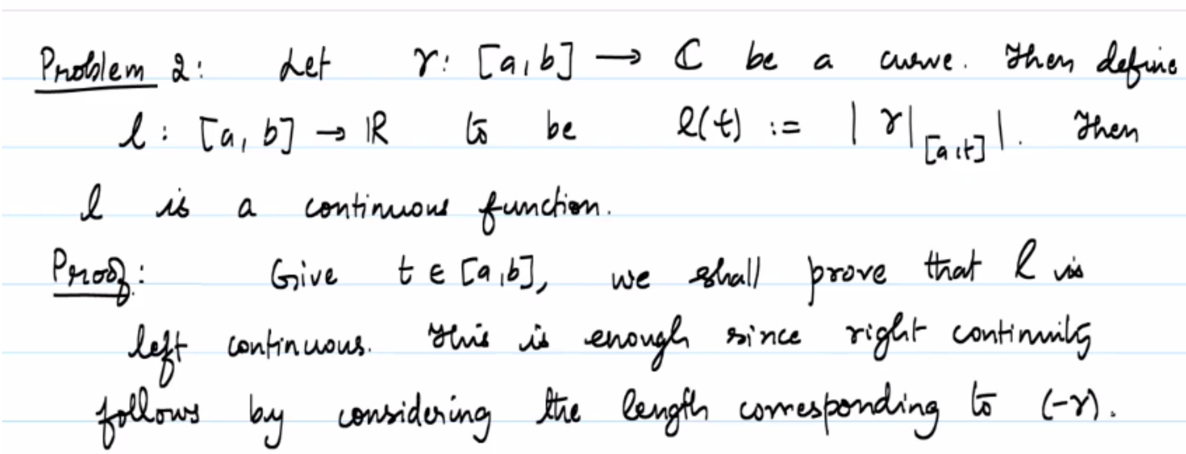
Then,

$$
\begin{aligned}
\Big||\gamma_1+\gamma_2|-\big(|\gamma_1|+|\gamma_2|\big)\Big| &\le \left||\gamma_1+\gamma_2|-\sum_{j=1}^{n}\big|(\gamma_1+\gamma_2)(t_j)-(\gamma_1+\gamma_2)(t_{j-1})\big|\right| \\
&\quad+\left||\gamma_1|+|\gamma_2|-\sum_{j=1}^{n}\big|(\gamma_1+\gamma_2)(t_j)-(\gamma_1+\gamma_2)(t_{j-1})\big|\right| \\
&<\frac{\epsilon}{3}+\left||\gamma_1|-\sum_{j=1}^{k}\big|\gamma_1(t_j)-\gamma_1(t_{j-1})\big|\right|+\left||\gamma_2|-\sum_{j=k+1}^{n}\big|\gamma_2(t_j)-\gamma_2(t_{j-1})\big|\right| \\
&<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon
\end{aligned}
$$

Since ϵ that was chosen was arbitrary, we have

$$|\gamma_1+\gamma_2|=|\gamma_1|+|\gamma_2|.$$

(Refer Slide Time: 11:26)



PROBLEM 2. Let $\gamma:[a,b]\longrightarrow\mathbb{C}$ be a curve. Define $\ell:[a,b]\longrightarrow\mathbb{R}$ to be $\ell(t):=\big|\gamma\restriction_{[a,t]}\big|$. Then ℓ is a continuous function.

SOLUTION 2. Given $t\in[a,b]$, we shall prove that ℓ is left continuous at t. This is enough since right continuity follows by considering the length corresponding to $(-\gamma)$.

Let $\epsilon > 0$ be given. Since γ is uniformly continuous on $[a,b]$, $\exists \delta'$ such that

$$|\gamma(s) - \gamma(t)| < \frac{\epsilon}{2}$$

whenever $|t - s| < \delta'$.

Let $P: \ a = t_0 < t_1 < \cdots < t_n = t$ be such that

$$\left|\gamma \restriction_{[a,t]}\right| - \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| < \frac{\epsilon}{2}.$$

(Refer Slide Time: 16:20)

Let δ < min $(\delta', |t_n - t_{n-1}|)$ and $t' \in (t-\delta, t]$.

By considering the refinement

$a = t_0 < t_1 < \dots < t_n = t' < t_{n+1} = t$.

$$|\gamma|_{[a,t]}| - \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| < \epsilon/2.$$

Let $\delta < \min(\delta', |t_n - t_{n-1}|)$ and $t' \in (t - \delta, t]$.

By considering the refinement, $a = t_0 < t_1 < \cdots < t_n = t' < t_{n+1} = t$,

$$\left|\gamma \restriction_{[a,t]}\right| - \sum_{j=1}^{n+1} \left|\gamma(t_j) - \gamma(t_{j-1})\right| < \frac{\epsilon}{2}.$$

Hence,

$$\left(\left|\gamma \restriction_{[a,t']}\right| + \left|\gamma \restriction_{[t',t]}\right|\right) - \left(\sum_{j=1}^{n} \left|\gamma(t_j) - \gamma(t_{j-1})\right|\right) - \left|\gamma(t) - \gamma(t')\right| < \frac{\epsilon}{2}.$$

Now regrouping the above equation,

$$\left(\left|\gamma \restriction_{[a,t']}\right| - \sum_{j=1}^{n} \left|\gamma(t_j) - \gamma(t_{j-1})\right|\right) + \left(\left|\gamma \restriction_{[t',t]}\right| - \left|\gamma(t) - \gamma(t')\right|\right) < \frac{\epsilon}{2}.$$

Since $\left|\gamma \restriction_{[a,t']}\right| > \sum_{j=1}^{n} \left|\gamma(t_j) - \gamma(t_{j-1})\right|$, we have $\left|\gamma \restriction_{[a,t']}\right| - \sum_{j=1}^{n} \left|\gamma(t_j) - \gamma(t_{j-1})\right| = c > 0$.

Thus,

$$\left|\gamma\restriction_{[t',t]}\right| - \left|\gamma(t)-\gamma(t')\right| < \frac{\epsilon}{2} - c < \frac{\epsilon}{2}.$$

Hence,

$$\left|\gamma\restriction_{[t',t]}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore,

$$\left|\gamma\restriction_{[t',t]}\right| < \epsilon.$$

$$\implies \left|\gamma\restriction_{[a,t]}\right| - \left|\gamma\restriction_{[a,t']}\right| < \epsilon.$$

That is,

$$\ell(t) - \ell(t') < \epsilon.$$

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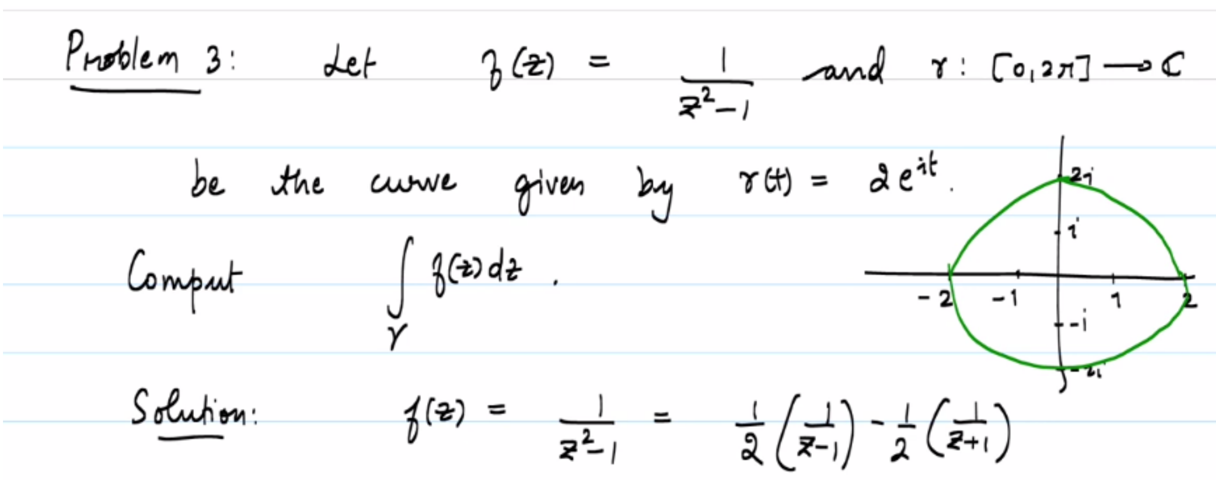

Problem 3. Let $f(z) = \dfrac{1}{z^2-1}$ and $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ be the curve given by $\gamma(t) = 2e^{it}$. Compute

$$\int_\gamma f(z)dz.$$

SOLUTION 3. We can decompose f into partial fractions,

$$f(z) = \frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1}\right) - \frac{1}{2}\left(\frac{1}{z+1}\right).$$

Since $\gamma(t) = 2e^{it}$, we have

$$\begin{aligned}\int_\gamma f(z)dz &= \frac{1}{2}\int_\gamma \frac{1}{z-1}dz - \frac{1}{2}\int_\gamma \frac{1}{z+1}dz \\ &= \frac{1}{2}\int_0^{2\pi} \frac{2ie^{it}}{(2e^{it}-1)}dt - \frac{1}{2}dt\int_0^{2\pi} \frac{2ie^{it}}{2e^{it}+1}dt\end{aligned}$$

(Refer Slide Time: 27:57)

Consider

$$\frac{1}{2}\int_0^{2\pi} \frac{2ie^{it}dt}{(2e^{it}-1)} = i\int_0^{2\pi} \frac{e^{it}\overbrace{(2e^{-it}-1)}^{(2-e^{it})}}{\underbrace{|2e^{it}-1|^2}_{(2\cos t-1)^2+(2\sin t)^2}} dt.$$

$$= i\int_0^{2\pi} \frac{(2-\cos t) - i\sin t}{5-4\cos t} dt.$$

Consider $\frac{1}{2}\int_0^{2\pi} \frac{2ie^{it}}{(2e^{it}-1)}dt$. Since $2e^{it}-1 = 2\cos t - 1 + i2\sin t$, we have

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi} \frac{2ie^{it}}{(2e^{it}-1)}dt &= i\int_0^{2\pi} \frac{e^{it}\left(2e^{-it}-1\right)}{\left|2e^{it}-1\right|^2}dt \\ &= i\int_0^{2\pi} \frac{(2-\cos t) - i\sin t}{5-4\cos t}dt \\ &= \int_0^{2\pi} \frac{\sin t}{5-4\cos t}dt + i\int_0^{2\pi} \frac{(2-\cos t)}{5-4\cos t}dt.\end{aligned}$$

Now by computing the integral of real valued functions on the RHS, we will get,

$$\int_0^{2\pi} \frac{\sin t}{5-4\cos t}dt = 0$$

and

$$\int_0^{2\pi} \frac{(2-\cos t)}{5-4\cos t} dt = \pi.$$

Hence,

$$\frac{1}{2}\int_\gamma \frac{1}{z-1} dz = \frac{1}{2}\int_0^{2\pi} \frac{2ie^{it}}{(2e^{it}-1)} dt = i\pi.$$

By a similar computation,

$$\frac{1}{2}\int_\gamma \frac{1}{z+1} dz = i\pi.$$

Hence,

$$\int_\gamma \frac{1}{z^2-1} dz = i\pi - i\pi = 0.$$

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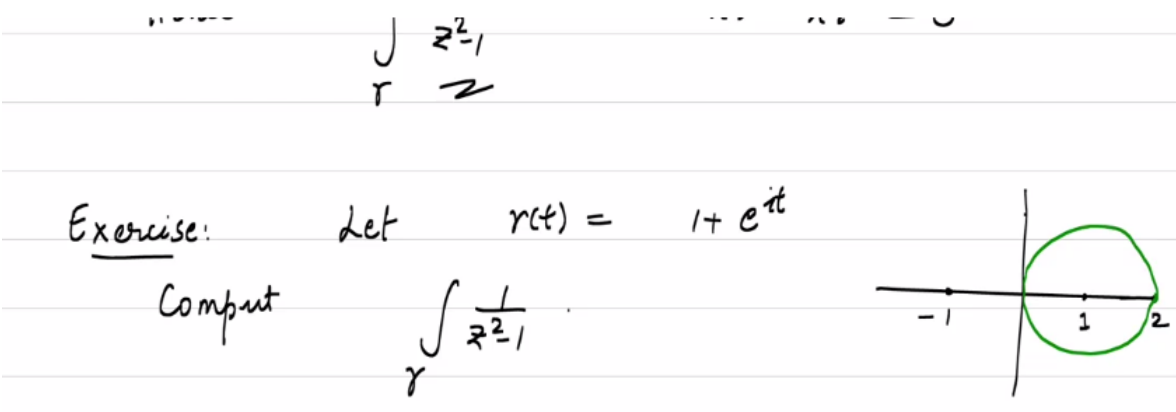

EXERCISE 4. Let $\gamma(t) = 1 + e^{it}$. Compute

$$\int_\gamma \frac{1}{z^2-1} dz.$$

PROBLEM 5. Prove that the function $f(z) = \dfrac{1}{z}$ does not have an anti-derivative in $\mathbb{C} \setminus \{0\}$.

SOLUTION 4. Let us consider $\gamma(t) = e^{it}$ on $[0, 2\pi]$.

If $f(z)$ had an anti-derivative in $\mathbb{C} \setminus \{0\}$, say F, then

$$\int_\gamma f(z) dz = F(1) - F(1) = 0.$$

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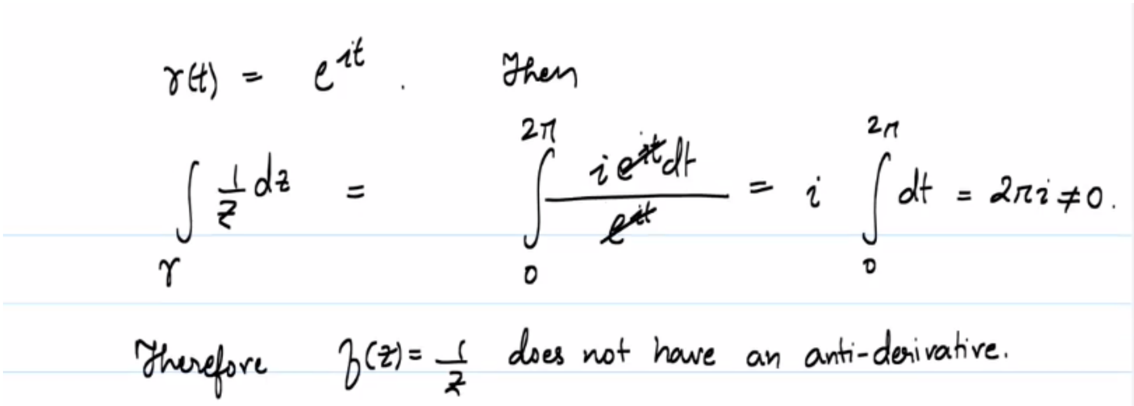

Let us compute $\int_\gamma f(z)dz$.

$$\int_\gamma \frac{1}{z} dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = i \int_0^{2\pi} dt = 2\pi i \neq 0.$$

Therefore $f(z) = \dfrac{1}{z}$ does not have an anti-derivative.

PROBLEM 6. Let $\Omega \subseteq \mathbb{C}$ be an open subset of $\mathbb{C}$ and $f, g : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Let $\gamma : [a, b] \longrightarrow \Omega$ be a rectifiable curve. Then

$$\int_\gamma f(z)g'(z)dz = f(z_1)g(z_1) - f(z_0)g(z_0) - \int_\gamma f'(z)g(z)dz.$$

where z_0 and z_1 are the initial and terminal point of γ respectively.

SOLUTION 5. Let $F(z) = (f(z)g(z))'$. By the fundamental theorem of calculus,

$$\int_\gamma F(z)dz = f(z_1)g(z_1) - f(z_0)g(z_0) \tag{4}$$

(Refer Slide Time: 42:27)

$$\int_\gamma F(z)\,dz = f(z_1)g(z_1) - f(z_0)g(z_0).$$

By the product rule $F(z) = f'(z)g(z) + f(z)g'(z)$.

$$\int_\gamma F(z)\,dz = \int_\gamma f'(z)g(z)\,dz + \int_\gamma f(z)g'(z)\,dz$$

By product rule, $F(z) = f'(z)g(z) + f(z)g'(z)$.

$$\int_\gamma F(z)dz = \int_\gamma f'(z)g(z)dz + \int_\gamma f(z)g'(z)dz. \tag{5}$$

By (4) and (5) , we have

$$\int_\gamma f(z)g'(z)dz = f(z_1)g(z_1) - f(z_0)g(z_0) - \int_\gamma f'(z)g(z)dz.$$