Complex Analysis
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Lecture – 2
Conjugation and Absolute Value

In the last lecture, we constructed an explicit field of complex numbers, a field which contains $\mathbb{R}$ as a subfield and a root i to the polynomial $x^2 + 1$. We also showed that if there is a subfield of the field of complex numbers which contains $\mathbb{R}$ and i, it should necessarily be the entire set $\mathbb{C}$. We also showed a uniqueness up to field isomorphism of the field of complex numbers, so hence we can talk about the field of complex numbers here.

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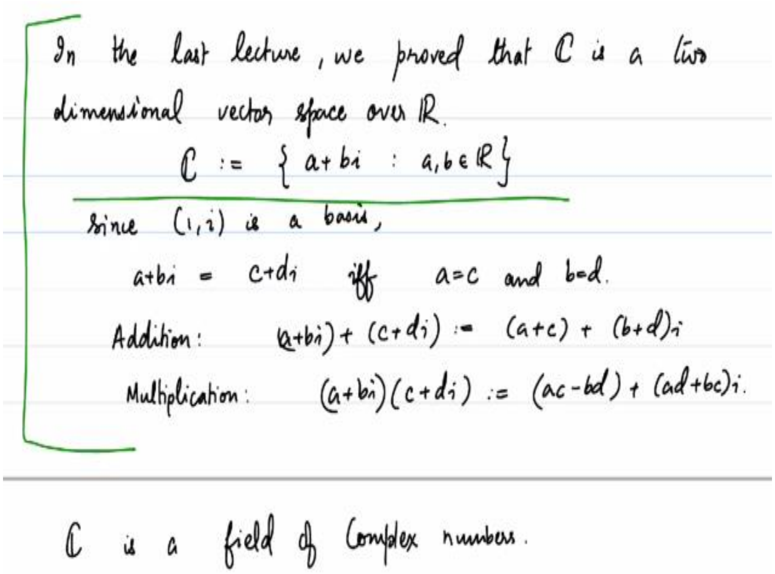

In the last lecture, we proved that $\mathbb{C}$ is a two dimensional vector space over $\mathbb{C}$. We, in fact, get hold of an explicit basis of $\mathbb{C}$ over $\mathbb{R}$, which is $\{1, i\}$.

Let us define $\mathbb{C} \coloneqq \{a + ib : a, b \in \mathbb{R}\}$ and since $\{1, i\}$ is a basis, $a + ib = c + id$ if and only if $a = c, b = d$. We even have an explicit expression for the addition and multiplication. So the addition of two complex numbers is given by, $(a + ib) + (c + id) = (a + c) + i(b + d)$ and we also defined the multiplication as being the product of cosets, that is, $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$. If you have not seen a course on abstract algebra or ring theory you may as well start the course, this course complex analysis here by taking this as our definition of field of complex numbers. It is a collection of all elements of the type $a + ib$ where a and b are real numbers.

Addition and multiplication are defined as above and you can check that $\mathbb{C}$ is a field of complex numbers with these operations.

We know that any two dimensional vector space over $\mathbb{R}$ going to be isomorphic as vector spaces to $\mathbb{R}^2$.

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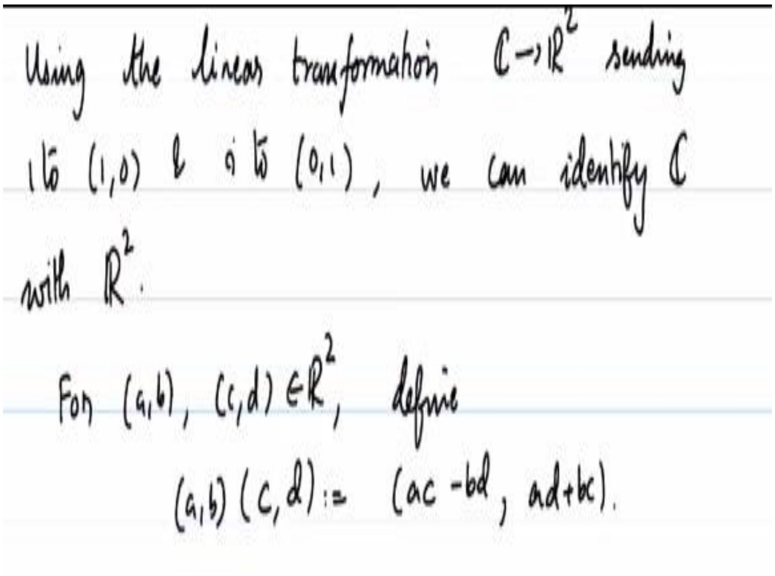

By looking at the linear transformation from $\mathbb{C} \to \mathbb{R}$ which sends 1 to (1, 0) and i to (0, 1), we could also define a field structure on $\mathbb{R}^2$ and we can identify $\mathbb{C}$ with $\mathbb{R}^2$.

Now we can define multiplication on $\mathbb{R}^2$. For $(a, b), (c, d) \in \mathbb{R}^2$, define $(a, b)(c, d) = (ac - bd, ad + bc)$. This is the definition for the multiplication in $\mathbb{R}^2$ and this particular definition is something which we had obtained from considering the multiplication of cosets in $\mathbb{R}[x]/\langle x^2 + 1\rangle$.

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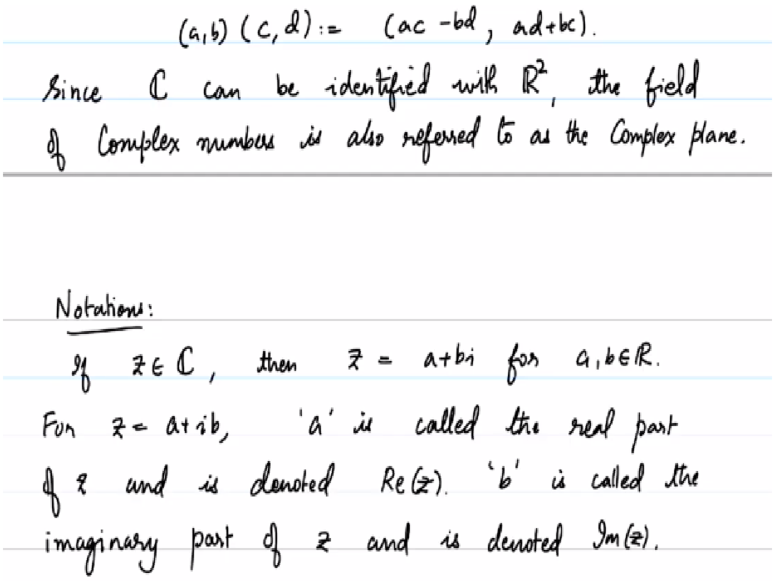

In this course if z is a complex number then $z = a + ib$ for $a, b \in \mathbb{R}$. Then a is called the real part of z and is denoted $\mathfrak{Re}(z)$ and at the same time b is called the imaginary part of z and is denoted $\mathfrak{Im}(z)$.

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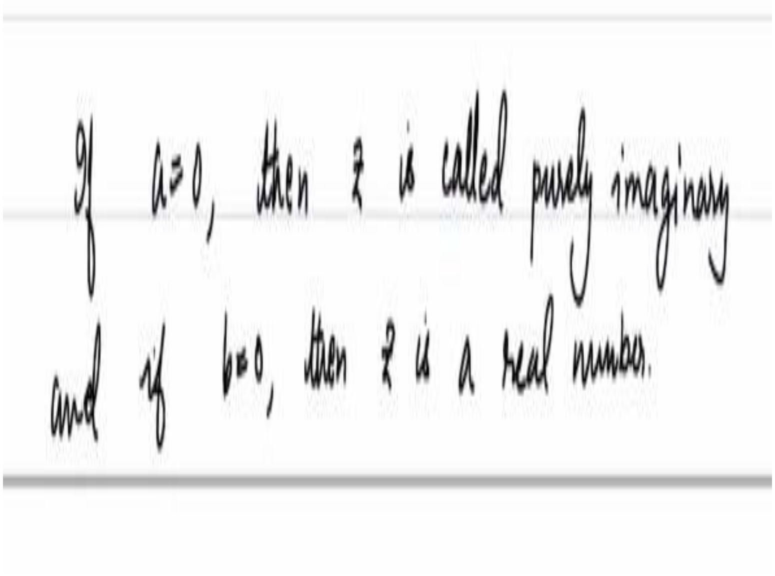

If $a = 0$, then z is called purely imaginary and if $b = 0$ it is a real then z is purely real or is a real number.

The vector space$\mathbb{R}^2$ is something which you might be very familiar with, it is the plane which we are usually accustomed to and we are familiar with geometry there and because $\mathbb{C}$ can be identified with $\mathbb{R}^2$, the field of complex numbers is also sometimes referred to as the complex plane.

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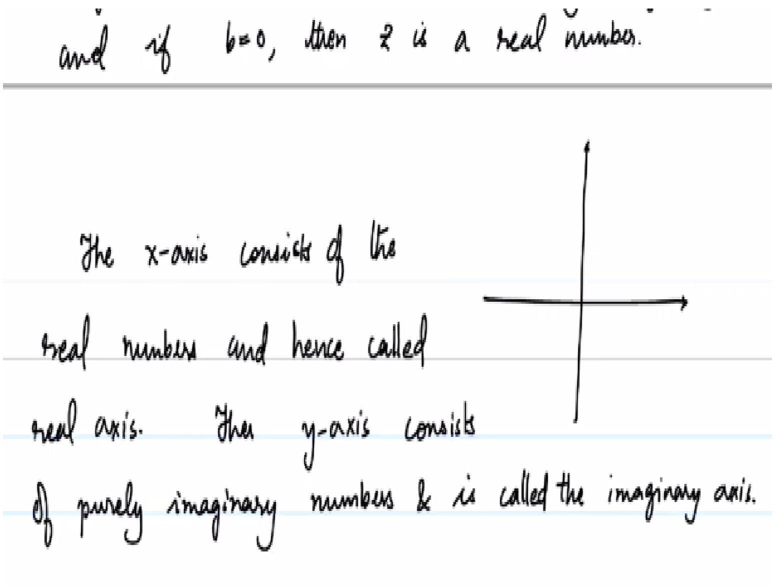

We have the x-axis and the y-axis on $\mathbb{R}^2$ and since $\mathbb{C}$ can be identified with $\mathbb{R}^2$ by the linear transformation sending 1 to (1, 0) and therefore the x-axis consists of real numbers and is called the real axis. Hence when we say real axis, we mean the x-axis.

And in $a + ib$, if a is 0, then ib will be mapped to $(0, b)$ and that will be the y-axis. The y-axis consists of purely imaginary numbers and is called the imaginary axis.

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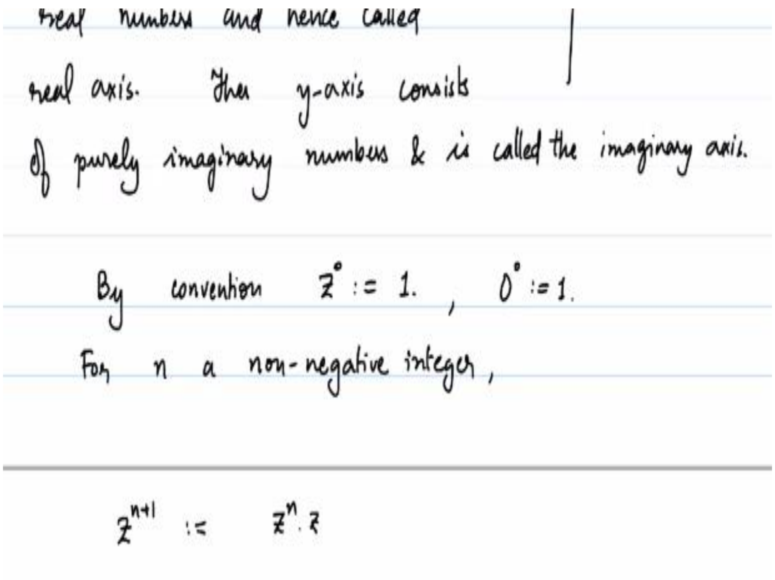

By convention for any complex number $z, z^0 \coloneqq 1$ and also $0^0 := 1$. For n, a non-negative integer, define $z^{n+1} = z \cdot z^n$, assuming that z^n has been already defined.

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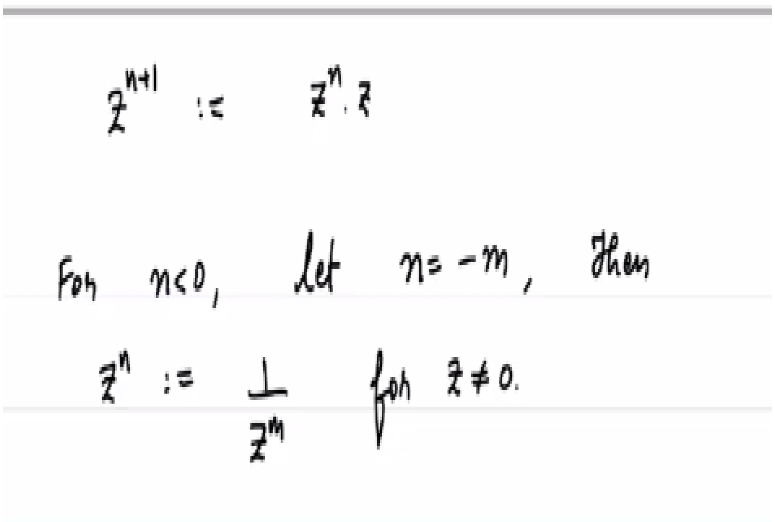

For $n < 0$, let $n = -m, m > 0$, then for $z \neq 0, z^n \coloneqq \frac{1}{z^m}$.

These are the usual notations which we have borrowed from field theory that is how we will define the exponentiation. In this course, we will be defining exponents with arbitrary exponents, but we will come to that later.

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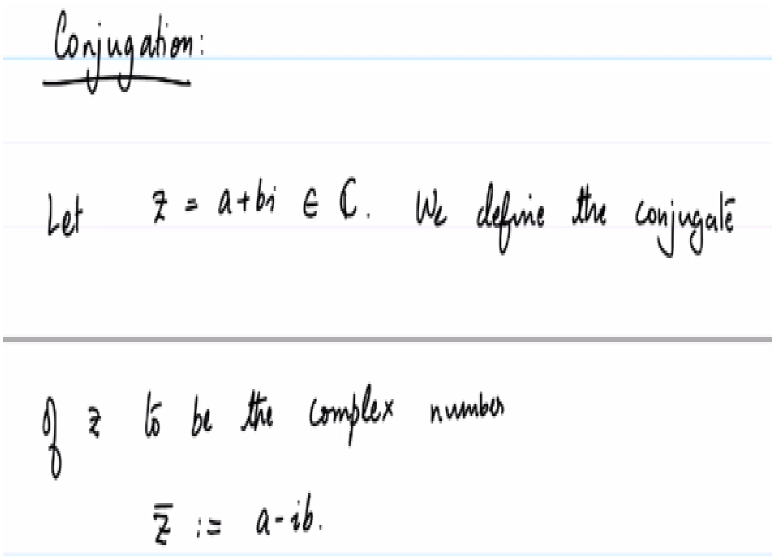

Let us next introduce the important concept of conjugation. Let $z = a + ib \in \mathbb{C}$. We define the conjugate of z to be the complex number $\bar{z} \coloneqq a - ib$.

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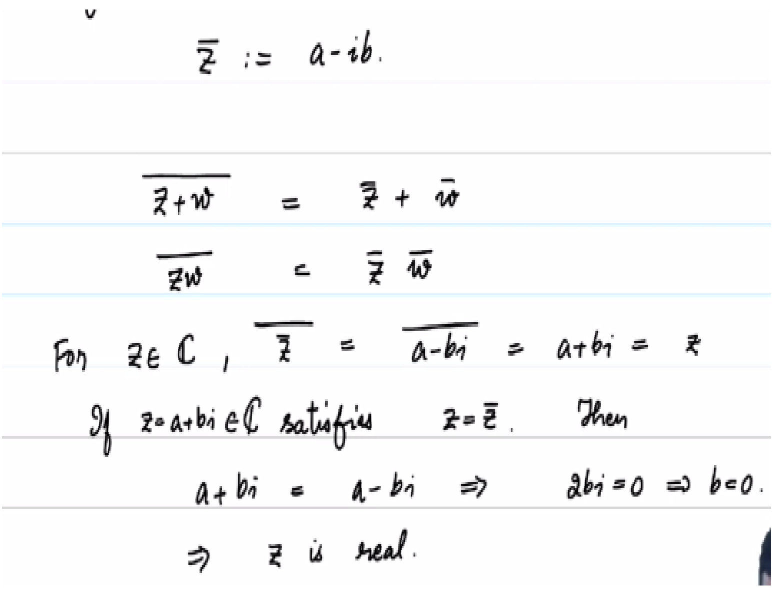

It is immediate from the definition that $\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$, the proof of which is left as an exercise. You could write it down explicitly in terms of coordinates, compute both left hand side and right hand side and see that they do match. If $z = a + ib \in \mathbb{C}$ satisfies $z = \bar{z}$,then $z = \bar{z} \Rightarrow a + ib = a - ib \Rightarrow 2ib = 0 \Rightarrow b = 0$.

Since 1 and i are forming a basis, $b = 0$ tells us that z is a real. If that is real, then naturally

$b = 0$, and therefore$z = \bar{z}$.

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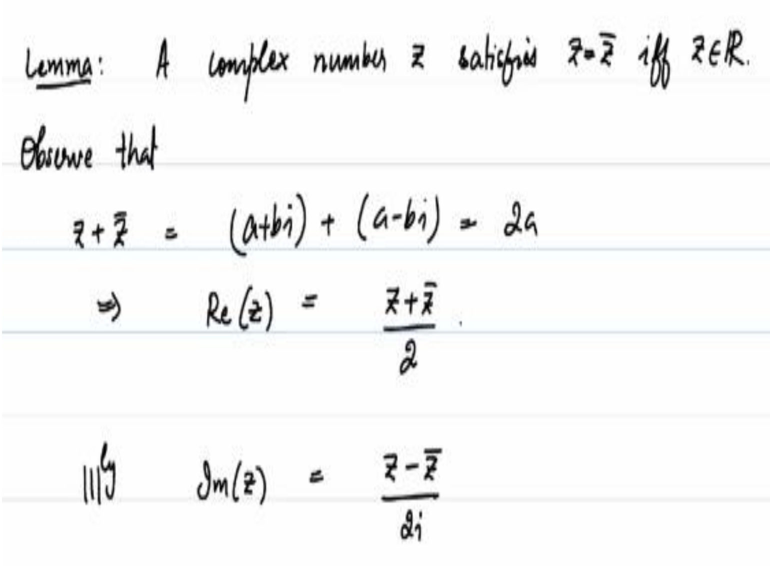

Lemma: A complex number z satisfies $z = \bar{z}$ if and only if $z \in \mathbb{R}$.

Now observe that $z + \overline{z} = (a + ib) + (a - ib) = 2a \Rightarrow \mathfrak{Re}(z) = \frac{z+\bar{z}}{2}$. Similarly

$\mathfrak{Im}(z) = \frac{z-\bar{z}}{2i}$.

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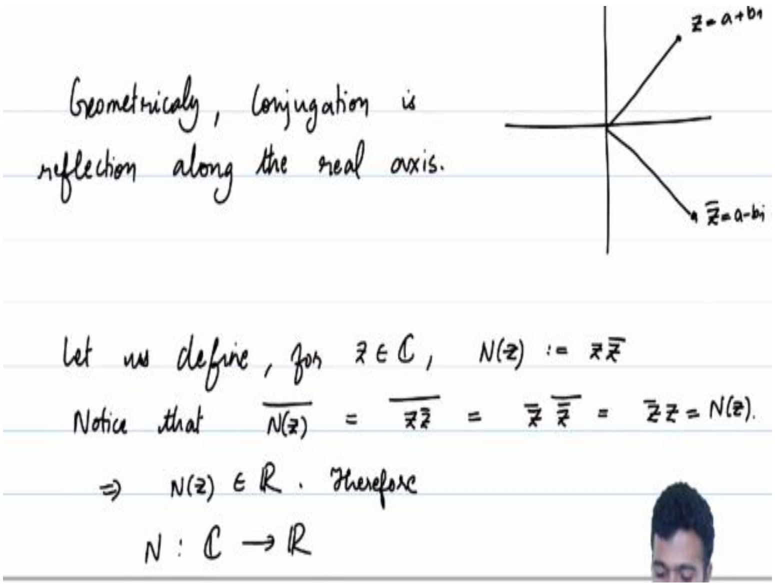

The conjugation is a notion which you will be familiar with from linear algebra. Geometrically, conjugation is the linear transformation which is the reflection along the real axis.

Now let us take the first step towards studying analysis and to do that let us try to see the topology on the field of complex numbers.

But before we explore all that, let us define a function. Let us take any complex number z, define$N(z) = z \cdot \bar{z}$. This is the function from $\mathbb{C}$ to $\mathbb{C}$, but there are a few observations immediately, which we can make.
Notice that for $z \epsilon \mathbb{C}$, $\overline{N(z)} = \overline{z \cdot \bar{z}} = \bar{z} \cdot \bar{\bar{z}} = \bar{z} \cdot z = N(z)$

Then we just observed a few minutes back that a complex number is equal to its conjugate if and only if it is a real number. This implies that $N(z)$ belongs to $\mathbb{R}$ and therefore $N : \mathbb{C} \longrightarrow \mathbb{R}$.

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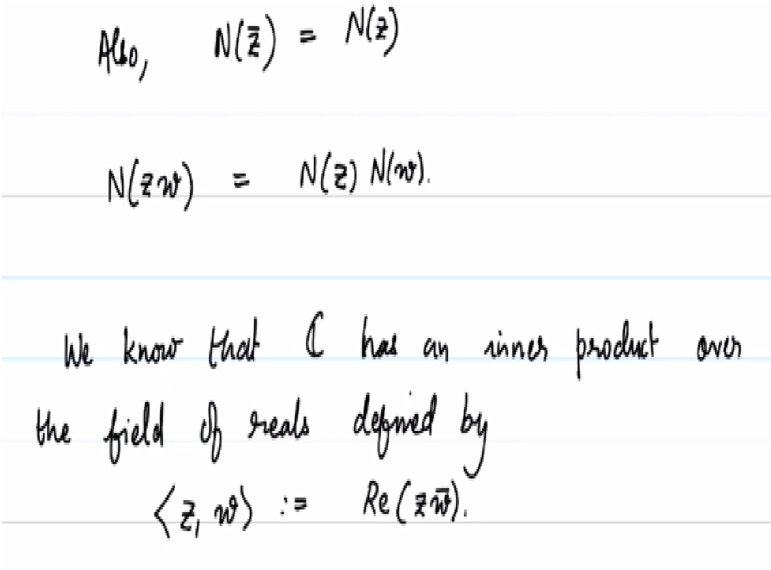

Also check that $N(\bar{z}) = N(z)$ and $N(z \cdot w) = N(z) \cdot N(w)$. This function is many times referred to as the field norm and it is an algebraic concept. It is a field norm of $\mathbb{C}$ over $\mathbb{R}$.
Our next goal would be to explicitly get hold of the metric with which will be working on the field of complex numbers. For that for the linear algebra we can use the fact that $\mathbb{C}$ is vector space over $\mathbb{R}$ and we know that $\mathbb{C}$ has an inner product over $\mathbb{R}$ defined by $\langle z, w \rangle := \mathfrak{Re}(z\bar{w})$.

This is the standard inner product which we generally consider on $\mathbb{C}$ considered over the field of real numbers.

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$$\langle z, w \rangle := Re(z\bar{w}).$$

With this inner product, $\mathbb{C}$ is a real inner product

space.

With this inner product, $\mathbb{C}$ is an inner product space. You must check explicitly that this defines an inner product on $\mathbb{C}$ over $\mathbb{R}$ by checking all the properties of an inner product.

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This inner product gives us the following norm

$$|z| = \langle z, z \rangle^{1/2} = \left(Re(z\bar{z})\right)^{1/2}$$
$$= \left(Re(N(z))\right)^{1/2}$$
$$= N(z)^{1/2}.$$

This inner product can be used to define a norm given by $|z| = \langle z, z\rangle^{1/2} = (\Re\mathrm{e}(z \cdot \bar{z}))^{1/2}$, but we have already defined $N(z) = z \cdot \bar{z}$ and hence $|z| = [\Re\mathrm{e}(N(z))]^{1/2}$. Since we have proved that $N(z)$ is a real number $\Re\mathrm{e}(N(z)) = N(z)$. Hence $|z| = N(z)^{1/2}$.

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If $z = a + bi$

$$N(z) = (a+bi)(a-bi) = a^2+b^2$$

$$|z| = (a^2+b^2)^{1/2}.$$

Now let us see what happens in coordinates. If $z = a + ib, N(z) = (a + ib)(a - ib) = a^2 + b^2$. Then we can define norm in terms of the coordinates of z as $|z| = (a^2 + b^2)^{1/2}$

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If z is real, then $b=0$, (i.e. $z=a$)

$$\Rightarrow |z| = (a^2)^{1/2} = |a|$$

Therefore the norm which we defined naturally extends the absolute value on $\mathbb{R}$ and hence the norm on $\mathbb{C}$ will also be called the absolute value on $\mathbb{C}$.

If z is real, then $b = 0$ and this would imply that the norm of $z, |z| = (a^2 + 0)^{1/2} = |a|$. Therefore, the norm which we just defined naturally extends the absolute value which we are familiar with on the field of real numbers and hence the norm on $\mathbb{C}$ will also be called the absolute value.

In any normed vector space, we can talk about a metric.

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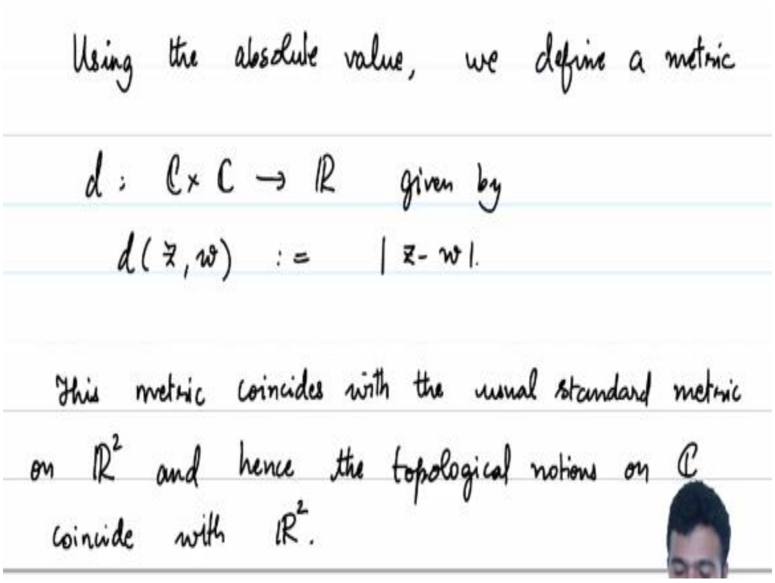

Let us now define a metric on this particular normed vector space. In other words, let us define a metric on $\mathbb{C}$ using the norm which we just defined. We define a metric $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by, for $z, w \epsilon \mathbb{C}, d(z, w) = |z - w|$. This will turn out to be a metric on $\mathbb{C}$. I will leave it as an exercise for you to check that this indeed coincides with the standard metric on $\mathbb{R}^2$.

The standard metric on $\mathbb{R}^2$ is $d\big((a, b), (c, d)\big) = \sqrt{(a - c)^2 + (b - d)^2}$.

Hence the basic topological notions of open sets, closed sets, boundary, convergence, limits, continuity all these things are the usual topological notions we are familiar with from a course on real analysis and hence the topological notions on $\mathbb{C}$ coincide with $\mathbb{R}^2$.

The metric d is defined using the absolute value which in turn is being defined using the field norm, which is an algebraic property.

If you go to different construction of the field of complex numbers, there will be a corresponding field norm there and we can do all these things again. So, basically the topology and the analysis that follows is going to be invariant under which choice of complex numbers you pick.