Complex Analysis

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Lecture No – 19

Second Fundamental Theorem of Calculus

In this lecture, we will be proving a second fundamental theorem of calculus, which is the complex analog of the second fundamental theorem of calculus, which you would have already seen in the real setting. But before we do that, let us recall the equivalent formulation of open connectedness in the complex plane.

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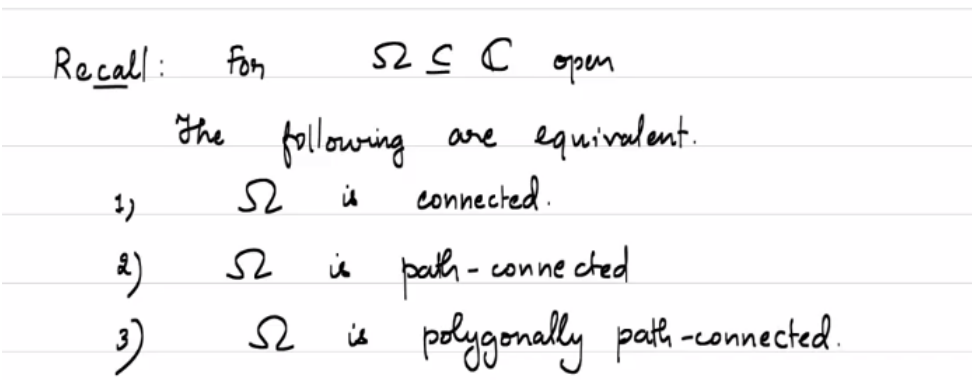

Recall: For $\Omega \subseteq \mathbb{C}$ open, the following are equivalent

i Ω is connected.

ii Ω is path-connected.

iii Ω is polygonally path-connected.

Polygonal path is the concatenation of straight line curves. $\gamma_{z_1 \to z_2 \to \cdots \to z_n}$ used to denote a polygonal path obtained by concatenating line joining z_j to z_{j+1}.

THEOREM 1 (Second Fundamental Theorem of Calculus). *Let $\Omega \subseteq \mathbb{C}$ be a non-empty open connected subset and let $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function such that*

$$\int_\eta f(z)\,dz = 0$$

whenever η is a closed polygonal path contained in Ω. Fix $z_0 \in \Omega$ and define

$$F(z_1) = \int_{\gamma} f(z)dz$$

where γ is a polygonal path from z_0 to z_1. Then F is well-defined holomorphic function such that $F'(z_1) = f(z_1)\ \forall z_1 \in \Omega$.

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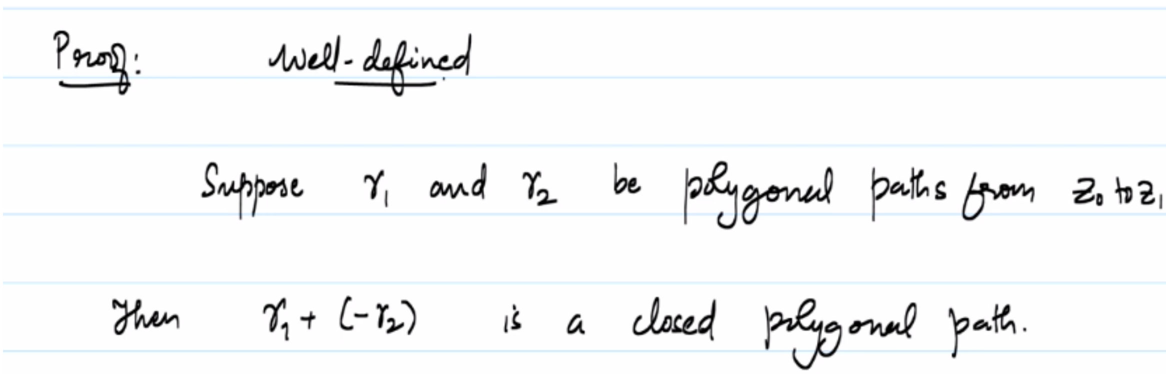

PROOF. First, let us prove that the F defined as in the theorem is a well defined function. That is it is independent of the polygonal path we choose.

Suppose γ_1 and γ_2 be polygonal paths from z_0 to z_1. Then $\gamma_1 + (-\gamma)$ is a closed polygonal path.

Hence,

$$0 = \int_{\gamma_1 + (-\gamma)} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{-\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz$$

$$\implies \int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Therefore $F(z_1)$ is well defined.

Let $z_1 \in \Omega$ and let $D(z_1, r) \subseteq \Omega$. Let γ be a polygonal path from z_0 to z_1.

Pick $z_2 \in D(z_1, r)$ and $\gamma_{z_1 \to z_2}$ be the straight line from z_1 to z_2. Then $\gamma + \gamma_{z_1 \to z_2}$ is a polygonal path from z_0 to z_2.

Now,

$$F(z_2) - F(z_1) = \int_{\gamma+\gamma_{z_1\to z_2}} f(z)dz - \int_{\gamma} f(z)dz$$
$$= \int_{\gamma_{z_1\to z_2}} f(z)dz$$

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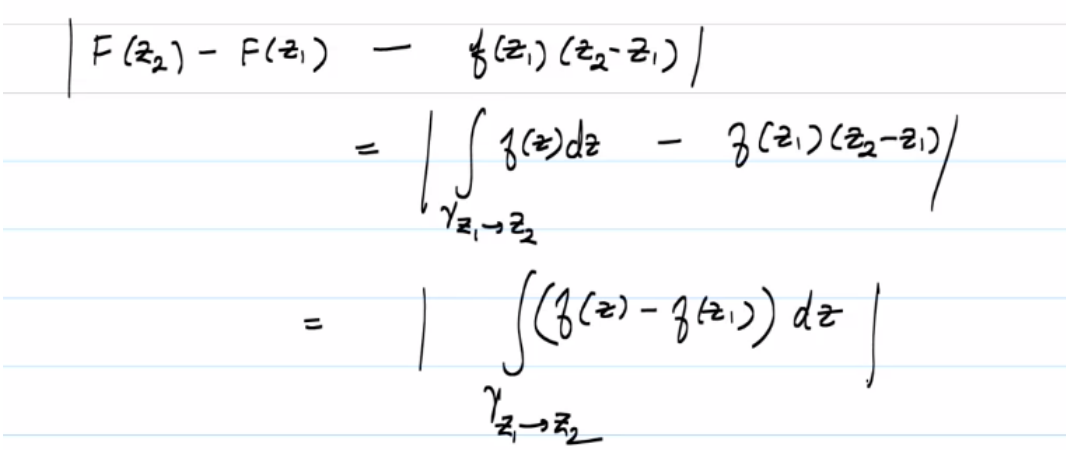

$$\left|F(z_2) - F(z_1) - f(z_1)(z_2 - z_1)\right| = \left|\int_{\gamma_{z_1\to z_2}} f(z)dz - f(z_1)(z_2 - z_1)\right|$$
$$= \left|\int_{\gamma_{z_1\to z_2}} \big(f(z) - f(z_1)\big)\, dz\right|.$$

Let $\epsilon > 0$ be given. Then by picking r small enough, we have

$$\left|F(z_2) - F(z_1) - f(z_1)(z_2 - z_1)\right| \le \epsilon |z_2 - z_1|.$$

Then,

$$\lim_{\substack{z_2\to z_1\\ z_2\in\Omega\setminus\{z_1\}}} \left|\frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1)\right| \le \epsilon.$$

Hence

$$\lim_{\substack{z_2\to z_1\\ z_2\in\Omega\setminus\{z_1\}}} \frac{F(z_2) - F(z_1)}{z_2 - z_1} = f(z_1)$$

That is,

$$F'(z_1) = f(z_1).$$

$\square$

PROPOSITION 2. *Let $\Omega \subseteq \mathbb{C}$ be a non-empty connected and $f : \Omega \longrightarrow \mathbb{C}$ be continuous function. Let $\gamma : [a,b] \longrightarrow \Omega$ be a rectifiable curve in Ω from z_1 to z_2. Then given $\epsilon > 0$, there exists a polygonal path $\sigma : [a,b] \longrightarrow \Omega$ from z_1 to z_2 such that*

$$\left| \int_\gamma f(z)dz - \int_\sigma f(z)dz \right| < \epsilon.$$

PROOF. We know that $\gamma([a,b])$ is compact set. Let $\mathscr{U} = \{D(z_0,r_0), D(z_1,r_1), \cdots, D(z_n.r_n)\}$ be a finite cover such that $\gamma([a,b]) \subset \bigcup\limits_{k=0}^{n} D(z_k,r_k)$ and $\overline{D(z_k,r_k)} \subset \Omega$ for each $k = 0,1,\cdots,n$. Let $K = \bigcup\limits_{k=0}^{n} \overline{D(z_k,r_k)}$. Then $K \subset \Omega$ and by Heine-Borel theorem, K is compact.

Since K is compact, f is uniformly continuous on K. Hence for given $\epsilon > 0$, we have $\delta > 0$ such that

$$\left| f(z) - f(w) \right| < \frac{\epsilon}{2|\gamma|}$$

whenever $|z - w| < \delta$ in K.

Let $P : a = t_0 < t_1 < \cdots < t_n = b$ be a partition such that $|P| < \min(\delta, \text{Lebesgue number of } \mathscr{U})$ and

$$\left| \int_\gamma f(z)dz - \sum_{j=1}^{n} f\left(\gamma(t_{j-1}\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right) \right| < \frac{\epsilon}{2}.$$

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Let σ be the polygonal path s.t.

$$\sigma\Big|_{[t_{j-1},t_j]} \equiv \gamma_{\gamma(t_{j-1})\to\gamma(t_j)}$$

$$\left|\int_\gamma f(z)dz - \int_\sigma f(z)dz\right| \leqslant$$

$$\left|\int_\gamma f(z)dz - \sum f(\gamma(t_{j-1}))(\gamma(t_j)-\gamma(t_{j-1}))\right|$$

Let σ be a polygonal path such that

$$\sigma\restriction_{[t_{j-1},t_j]} \equiv \gamma_{\gamma(t_{j-1})\to\gamma(t_j)}.$$

$$\begin{aligned}\left|\int_\gamma f(z)dz - \int_\sigma f(z)dz\right| \le &\left|\int_\gamma f(z)dz - \sum_{j=1}^n f\left(\gamma(t_{j-1})\right)\left(\gamma(t_j)-\gamma(t_{j-1})\right)\right| \\ &+\left|\sum_{j=1}^n f\left(\gamma(t_{j-1})\right)\left(\gamma(t_j)-\gamma(t_{j-1})\right) - \int_\sigma f(z)dz\right|\end{aligned}$$

$$\begin{aligned}\left|\int_\gamma f(z)dz - \int_\sigma f(z)dz\right| \le &\left|\int_\gamma f(z)dz - \sum_{j=1}^n f\left(\gamma(t_{j-1})\right)\left(\gamma(t_j)-\gamma(t_{j-1})\right)\right| \\ &+\left|\sum_{j=1}^n f\left(\gamma(t_{j-1})\right)\left(\gamma(t_j)-\gamma(t_{j-1})\right) - \int_\sigma f(z)dz\right|.\end{aligned}$$

Now,

$$\begin{aligned}
\left|\int_\sigma f(z)dz-\sum_{j=1}^n f\left(\gamma(t_{j-1})\left(\gamma(t_j)-\gamma(t_{j-1})\right)\right|\right. &= \left|\int_\sigma f(z)dz-\sum_{j=1}^n f\left(\gamma(t_{j-1})\left(\sigma(t_j)-\sigma(t_{j-1})\right)\right|\right. \\
&= \left|\int_\sigma f(z)dz-\sum_{j=1}^n \int_{\sigma\restriction_{[t_{j-1},t_j]}} f\left(\gamma(t_{j-1})\right)dz\right| \\
&= \left|\sum_{j=1}^n \int_{\sigma\restriction_{[t_{j-1},t_j]}} \left(f(z)-f\left(\gamma(t_{j-1})\right)\right)dz\right| \\
&\leq \sum_{j=1}^n \left|\int_{\sigma\restriction_{[t_{j-1},t_j]}} \left(f(z)-f\left(\gamma(t_{j-1})\right)\right)dz\right| \\
&\leq \sum_{j=1}^n \frac{\epsilon}{2|\gamma|}\left|\sigma(t_j)-\sigma(t_{j-1})\right| \\
&\leq \frac{\epsilon}{2|\gamma|}\sum_{j=1}^n \left|\gamma(t_j)-\gamma(t_{j-1})\right| \\
&\leq \frac{\epsilon}{2}.
\end{aligned}$$

Hence,

$$\left|\int_\gamma f(z)dz-\int_\sigma f(z)dz\right|<\epsilon.$$

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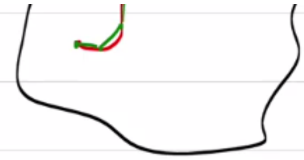

$$\left|\int_\gamma f(z)\,dz - \int_\sigma f(z)\,dz\right| \leq$$

$$\underbrace{\left|\int_\gamma f(z)\,dz - \sum f(\gamma(t_{j-1}))(\gamma(t_j)-\gamma(t_{j-1}))\right|}_{< \varepsilon/2}$$

$$+ \boxed{\left|\sum f(\gamma(t_{j-1}))(\gamma(t_j)-\gamma(t_{j-1})) - \int_\sigma f(z)\,dz\right|}$$

$$\boxed{\left|\int_\sigma f(z)\,dz - \sum_{j=1}^{n} f(\gamma(t_{j-1}))(\gamma(t_j)-\gamma(t_{j-1}))\right|}$$

□

Let us define two classes of curves, which will be of greatest interest. They are very special curves with a much better regularity.

DEFINITION (**Smooth Curves**). We say that a curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ is a smooth curve if γ is continuously differentiable and $\gamma'(t) \neq 0\ \forall\, t \in [a,b]$.

The straight line path, for example, is a smooth curve. The most of the curves we have actually encountered are smooth curves. In fact the equation of a circle is a smooth curve.

DEFINITION (**Contours**). We say that a curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ is a contour if it is the concatenation of finitely many smooth curves.

For example, the polygonal paths are contours, which are obtained by concatenation of finitely many straight line paths.