Complex Analysis

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Lecture No – 18

First Fundamental Theorem of Calculus

In this lecture, we will study the complex analog of the fundamental theorem of calculus, rather the first fundamental theorem of calculus. In a course on real analysis, you would have seen a version of the first fundamental theorem of calculus. in a course on real analysis, you would have seen a version of the first fundamental theorem of calculus. The complex analog is quite similar.

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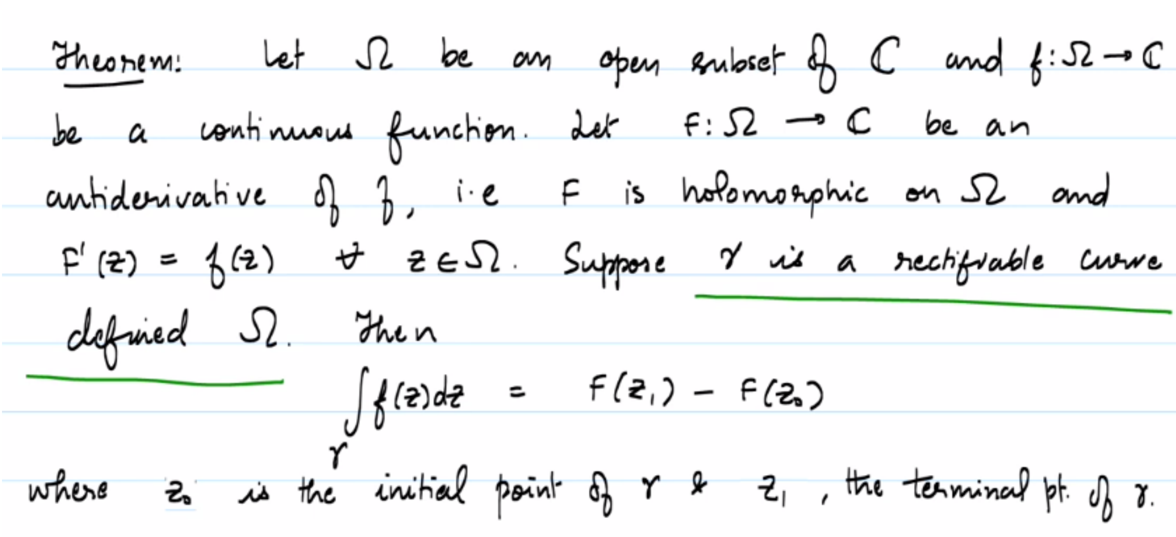

THEOREM 1. *Let Ω be an open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function. Let $F : \Omega \longrightarrow \mathbb{C}$ be an antiderivative of f, i.e., F is holomorphic on Ω and $F'(z) = f(z)\ \forall\, z \in \Omega$. Suppose γ is a rectifiable curve defined on Ω. Then*

$$\int_\gamma f(z)dz = F(z_1) - F(z_0)$$

where z_0 is the initial point of γ and z_1 is the terminal point of γ.

PROOF. Let $\gamma : [a,b] \longrightarrow \Omega$. Suppose γ is continuously differentiable. Then

$$\int_\gamma f(z)dz = \int_a^b f\left(\gamma(t)\right)\gamma'(t)dt.$$

Since $F'(z) = f(z) \forall z \in \Omega$,

$$\int_\gamma f(z)dz = \int_a^b F'\left(\gamma(t)\right)\gamma'(t)dt.$$

Now it is left as an exercise to the reader to check that

$$F'\left(\gamma(t)\right)\gamma'(t) = \left(F \circ \gamma\right)'(t)$$

which follows from chain rule and Cauchy-Riemann equations.
Hence,

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b \left(F \circ \gamma\right)'(t)dt \\ &= F\left(\gamma(b)\right) - F\left(\gamma(a)\right) \\ &= F(z_1) - F(z_0).\end{aligned}$$

So if the curve γ is continuously differentiable, our work becomes extremely easy and we will be able to conclude the complex analog of fundamental theorem of calculus from the fundamental theorem of calculus proved in the real analysis setting. However, we have put more generality here. We are only assuming that our curve is rectifiable. So we will have to work a bit more to prove our result in the complex analog case.

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Given $\varepsilon > 0$, let

$$\mathcal{U}_\varepsilon = \left\{ t \in [a,b] : \left| \int_a^{t_0} f(z)dz - (F(\gamma(t)) - F(\gamma(a))) \right| \le \varepsilon |\gamma|_{[a,t_0]} \ \forall\, a \le t_0 \le t \right\}$$

Enough to show that $\mathcal{U}_\varepsilon$ is both open and closed.

Given $\epsilon > 0$, let

$$\Omega_\epsilon = \left\{ t \in [a,b] : \left| \int_{\gamma\restriction_{[a,t_0]}} f(z)dz - \left(F\left(\gamma(t_0)\right) - F\left(\gamma(a)\right)\right) \right| \le \epsilon |\gamma \restriction_{[a,t_0]}|, \ \forall a \le t_0 \le t \right\}.$$

It is enough to show that Ω_ϵ is both open and closed.

Let t_0 be a limit point of Ω_ϵ, i.e., $\exists \{t_n\}_{n\in\mathbb{N}} \subset \Omega_\epsilon$ such that $t_n \to t_0$. If $t_n > t_0$, then $t_0 \in \Omega_\epsilon$.

Hence, we may assume that $t_n \uparrow t_0$.

$$\left| \int_{\gamma\restriction_{[a,t_0]}} f(z)dz - \left(F\left(\gamma(t_0)\right) - F\left(\gamma(a)\right)\right) \right|$$

$$= \left| \int_{\gamma\restriction_{[a,t_n]} + \gamma\restriction_{[t_n,t_0]}} f(z)dz - \left(F\left(\gamma(t_0)\right) - F\left(\gamma(t_n)\right) + F\left(\gamma(t_0)\right) - F\left(\gamma(a)\right)\right) \right|$$

$$\le \left| \int_{\gamma\restriction_{[a,t_n]}} f(z)dz - \left(F\left(\gamma(t_n)\right) - F\left(\gamma(a)\right)\right) \right| + \left| \int_{\gamma\restriction_{[t_n,t_0]}} f(z)dz - \left(F\left(\gamma(t_0)\right) - F\left(\gamma(t_n)\right)\right) \right|$$

$$\le \epsilon \left|\gamma \restriction_{[a,t_n]}\right| + M \left|\gamma \restriction_{[t_n,t_0]}\right| + \left|F\left(\gamma(t_0)\right) - F\left(\gamma(t_n)\right)\right|$$

Given $\epsilon' > 0$, we can pick t_n arbitrarily close to t_0 such that

$$\epsilon \left|\gamma \restriction_{[a,t_n]}\right| + M \left|\gamma \restriction_{[t_n,t_0]}\right| + \left|F\left(\gamma(t_0)\right) - F\left(\gamma(t_n)\right)\right| \le \epsilon \left|\gamma \restriction_{[a,t_n]}\right| + (M+1)\epsilon'.$$

Hence,

$$\left| \int_{\gamma\restriction_{[a,t_0]}} f(z)dz - \left(F\left(\gamma(t_0)\right) - F\left(\gamma(a)\right)\right) \right| \le \epsilon \left|\gamma \restriction_{[a,t_n]}\right| + (M+1)\epsilon'$$

Since ϵ' is arbitrary,

$$\implies \left| \int_{\gamma \restriction_{[a,t_0]}} f(z)\,dz - \left(F\left(\gamma(t_0)\right) - F\left(\gamma(a)\right)\right) \right| \le \epsilon \left| \gamma \restriction_{[a,t_0]} \right|.$$

Hence Ω_ϵ is closed.

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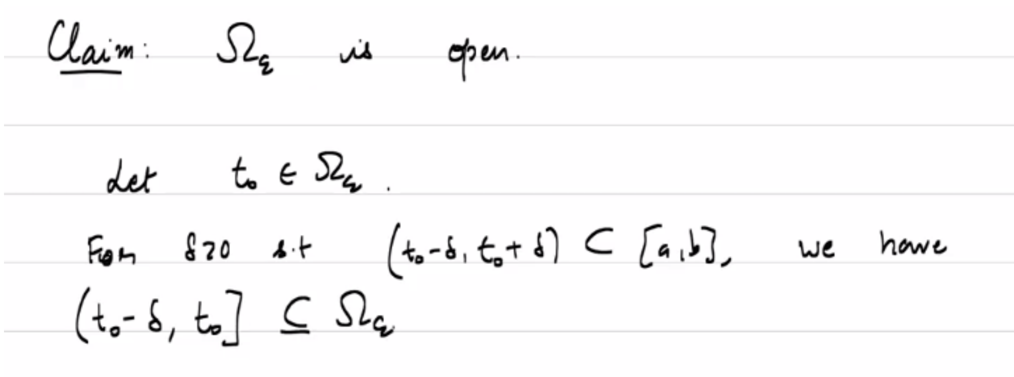

Claim: Ω_ϵ is open.

Let $t_0 \in \Omega_\epsilon$. For $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset [a, b]$, we have $(t_0 - \delta, t_0] \subseteq \Omega_\epsilon$.

For $t \in [t_0, t_0 + \delta)$.

Since F is holomorphic at $\gamma(t_0)$, given $\epsilon > 0, \exists \delta > 0$ such that

$$\left| \frac{F\left(\gamma(t)\right) - F\left(\gamma(t_0)\right)}{\gamma(t) - \gamma(t_0)} - f\left(\gamma(t_0)\right) \right| \frac{\epsilon}{2}$$

whenever $t \in (t_0, t_0 + \delta)$.

Then

$$\begin{aligned} \left| F\left(\gamma(t)\right) - F\left(\gamma(t_0)\right) - f\left(\gamma(t_0)\right)\left(\gamma(t) - \gamma(t_0)\right) \right| &\le \frac{\epsilon}{2} \left| \gamma(t) - \gamma(t_0) \right| \\ &\le \frac{\epsilon}{2} \left| \gamma \restriction_{[t_0,t]} \right|. \end{aligned}$$

Now,

$$f\left(\gamma(t_0)\right)\left(\gamma(t) - \gamma(t_0)\right) = \int_{\gamma \restriction_{[t_0,t]}} f\left(\gamma(t_0)\right) dz.$$

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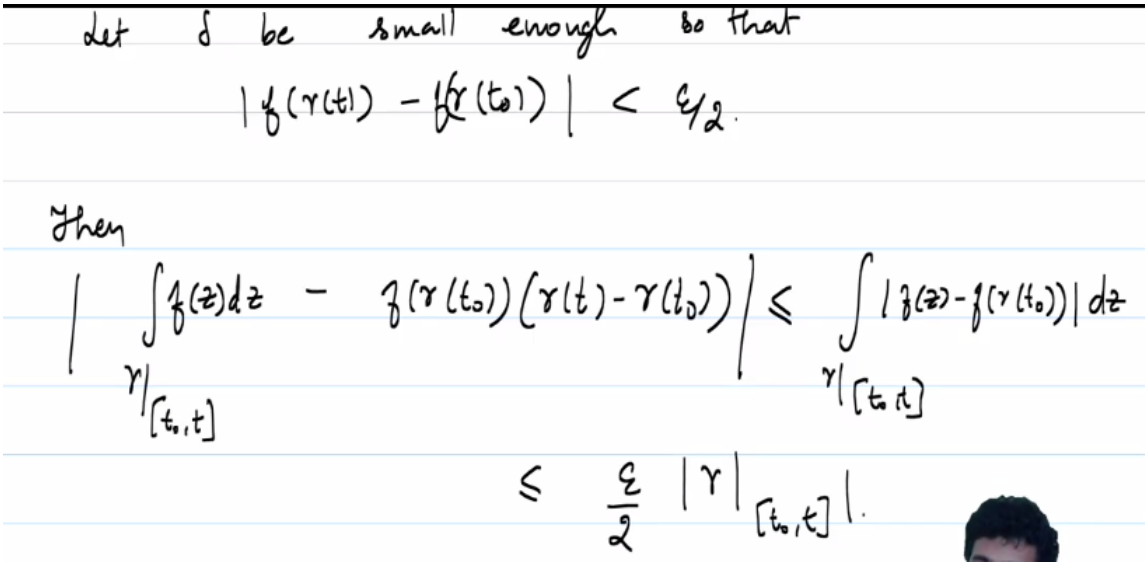

Let δ be small enough such that

$$\left|f\left(\gamma(t)\right)-f\left(\gamma(t_0)\right)\right|<\frac{\epsilon}{2}.$$

Then,

$$\begin{aligned}\left|\int_{\gamma\restriction_{[t_0,t]}} f(z)dz-f\left(\gamma(t_0)\right)\left(\gamma(t)-\gamma(t_0)\right)\right| &\leq \int_{\gamma\restriction_{[t_0,t]}}\left|f(z)-f\left(\gamma(t_0)\right)\right|dz\\ &\leq \frac{\epsilon}{2}\left|\gamma\restriction_{[t_0,t]}\right|.\end{aligned}$$

Hence by triangle inequality, we have

$$\left|\int_{\gamma\restriction_{[t_o,t]}} f(z)dz-\left(F\left(\gamma(t)\right)-F\left(\gamma(t_0)\right)\right)\right|\leq\epsilon\left|\gamma\restriction_{[t_0,t]}\right|\to(*).$$

We know that, since $t_0\in\Omega_\epsilon$, we have

$$\left|\int_{\gamma\restriction_{[a,t_o]}} f(z)dz-\left(F\left(\gamma(t_0)\right)-F\left(\gamma(a)\right)\right)\right|\leq\epsilon\left|\gamma\restriction_{[a,t_0]}\right|\to(**).$$

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Hence $\left| \int_{\gamma|_{[a,t]}} f(z)dz - \left(F(\gamma(t)) - F(\gamma(a))\right) \right| \leq \varepsilon \left| \gamma|_{[a,t_0]} \right|$

$$+ \varepsilon \left| \gamma|_{[t_0,t]} \right|.$$

$$\leq \varepsilon \left| \gamma|_{[a,t]} \right|.$$

$\forall \ t \in (t_0, t_0+\delta)$.

$\Rightarrow \quad (t_0-\delta, \ t_0+\delta) \subseteq \Omega_\varepsilon$.

Hence Ω_ε is open.

Hence,

$$\left| \int_{\gamma\restriction_{[a,t]}} f(z)dz - \left(F\left(\gamma(t)\right) - F\left(\gamma(a)\right)\right) \right| \leq \epsilon \left|\gamma\restriction_{[a,t_0]}\right| + \epsilon \left|\gamma\restriction_{[t_0,t]}\right|$$
$$\leq \epsilon \left|\gamma\restriction_{[t_0,t]}\right|.$$

This is true $\forall\, t \in (t_0, t_0+\delta) \implies (t_0-\delta, t_0+\delta) \subseteq \Omega_\epsilon$.

Hence Ω_ϵ is open.

Since $a \in \Omega_\epsilon$, we have by connectedness of $[a,b], \Omega_\epsilon = [a,b]$.

Therefore, $\forall\, \epsilon > 0$, we have

$$\left| \int_\gamma f(z)dz - \left(F\left(\gamma(b)\right) - F\left(\gamma(a)\right)\right) \right| \leq \epsilon|\gamma|.$$

$$\implies \int_\gamma f(z)dz = F(z_1) - F(z_0)$$

where $z_1 = \gamma(b)$ and $z_0 = \gamma(a)$.

□

EXAMPLE 2.

- We know that $\dfrac{de^z}{dz} = e^z$. Hence, if γ is a curve from z_1 to z_2, we have

$$\int_\gamma e^z dz = e^{z_2} - e^{z_1}.$$

- $\dfrac{d}{dz}\left(\dfrac{z}{2}\right) = z$. Then,

$$\int_\gamma z dz = \frac{z_2^2 - z_1^2}{2}$$

where z_1 and z_2 are initial and terminal point of γ respectively.

- On $\mathbb{C} \setminus \{0\}$, we have $\dfrac{d}{dz}\left(-\dfrac{1}{z}\right) = \dfrac{1}{z^2}$.

$$\int_\gamma \frac{1}{z^2}\, dz = \frac{1}{z_1} - \frac{1}{z_2}$$

where z_1 and z_2 are initial and terminal point of γ respectively.

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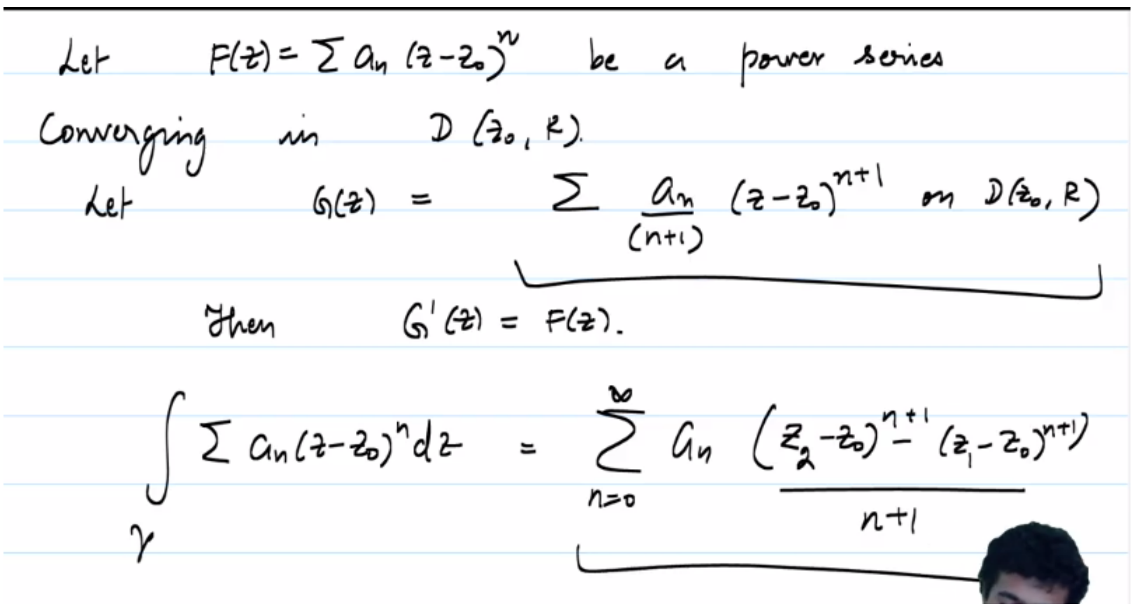

- Let $F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series converging in $D(z_0, R)$.

Let $G(z) = \sum_{n=0}^{\infty} \dfrac{a_n}{n+1}(z - z_0)^{n+1}$ on $D(z_0, R)$. Then $G'(z) = F(z)$.

$$\int_\gamma \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z_2 - z_0)^{n+1} - \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z_1 - z_0)^{n+1}.$$