Complex Analysis

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Lecture No – 17

Complex Integration Over Curves

In the last few lectures, we discussed the notion of complex differentiability in great detail. Just like in the case of real analysis, complex differentiability is tied together intricately with the notion of complex integration. When we say complex integration here, we mean integration of complex valued functions over rectifiable curves. We will define the integration of complex valued functions over rectifiable curves in this lecture. Thereafter, we will prove fundamental theorems of calculus for such functions. We will also later on will prove a special version of the fundamental theorem of calculus for holomorphic functions defined on nice domains, it is called the Cauchy's theorem. Let us recall the notion of the Riemann integral of real valued functions defined on closed interval $[a,b]$. We will be defining the complex line integral over rectifiable curves similarly.

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Recall that if $f : [a,b] \to \mathbb{R}$ be continous, then the Riemann integral of f over $[a,b]$ is the limit of the Riemann sums

$$\sum_{j=0}^{n} f(t_j^*)(t_j - t_{j-1})$$

Recall that if $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then the Riemann integral of f over $[a,b]$ is the limit of the Riemann sums

$$\sum_{j=1}^{n} f(t_j^*)(t_j - t_{j-1})$$

where the limit is taken as the partition size $|P|$ converges to 0.

Here $P : a = t_0 < t_1 < \cdots < t_n = b$ and $|P| = \max\limits_{1 \le j \le n} (t_j - t_{j-1})$ and $t_j^* \in [t_{j-1}, t_j]$.

This limit is denoted by $\displaystyle\int_a^b f(t)\,dt$.

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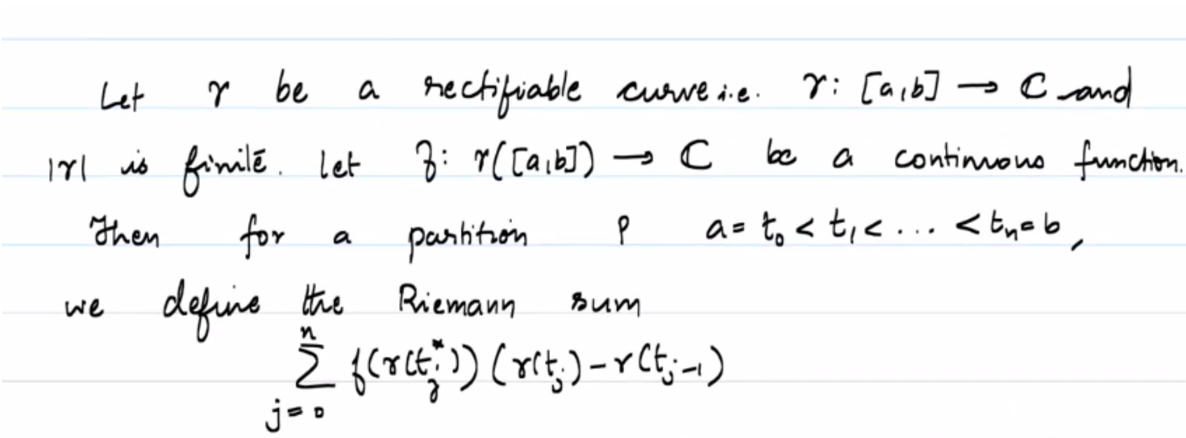

Let γ be a rectifiable curve. That is, $\gamma : [a,b] \longrightarrow \mathbb{C}$ continuous and $|\gamma|$ is finite. Let $f : \gamma([a,b]) \longrightarrow \mathbb{C}$ be a continuous function.

Then for a partition P of $[a,b]$, $a = t_0 < t_1 < \cdots < t_n = b$, we define the Riemann sum

$$\sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right)$$

where $t_J^* \in [t_{j-1}, t_j]$.

PROPOSITION 1. *The above Riemann sum converge to a complex limit as the partition size converge to 0. The limit is denoted by* $\displaystyle\int_\gamma f(z)\,dz$.
More precisely, given $\epsilon > 0$, $\exists\, \delta > 0$ *such that*

$$\left| \int_\gamma f(z)\,dz - \sum f\left(\gamma(t_j^*)\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right) \right| < \epsilon$$

whenever $|P| < \delta$.

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Proof: We shall prove that given $\epsilon > 0$, $\exists\, \delta > 0$ s.t if $P_1 : a = t_0 < t_1 < \cdots < t_n = b$ and $P_2 : a = s_0 < s_1 < \cdots s_m = b$ be partitions of size $< \delta$, then

$$\left| \sum_{j=0}^{n} f(\gamma(t_j^*))\,(\gamma(t_j) - \gamma(t_{j-1})) - \sum_{k=0}^{m} f(\gamma(s_k^*))\,(\gamma(s_k) - \gamma(s_{k-1})) \right| < \epsilon$$

PROOF. We shall prove that given $\epsilon > 0, \exists \delta > 0$ such that if $P_1 : a = t_0 < t_1 < \cdots < t_n = b$ and $P_2 : a = s_0 < s_1, \cdots < s_m = b$ be partitions of size less than δ, then

$$\left| \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right) - \sum_{k=1}^{m} f\left(\gamma(s_k^*)\right)\left(\gamma(s_k) - \gamma(s_{k-1})\right) \right| < \epsilon.$$

Let $R(P_1) = \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right)$ and $R(P_2) = \sum_{k=1}^{m} f\left(\gamma(s_k^*)\right)\left(\gamma(s_k) - \gamma(s_{k-1})\right)$.

If Q is a refinement of P_1, P_2, then $|Q| < \min(|P_1|, |P_2|)$.

$$|R(P_1) - R(P_2)| \le |R(P_1) - R(Q)| + |R(Q) - R(P_2)|.$$

Hence we may assume that P_2 is a refinement of P_1.

We thus have integers $0 = m_0 < m_1 < \cdots < m_n = m$ such that $t_j = s_{m_j}$.

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$$t_j = s_{m_j}$$

$$\left| \sum_{j=0}^{n} f(\gamma(t_j^*)) \left(\gamma(t_j) - \gamma(t_{j-1})\right) - \sum_{k=0}^{m} f(\gamma(s_k)) \left(\gamma(s_k) - \gamma(s_{k-1})\right) \right|$$

$$= \left| \qquad \nu \qquad - \sum_{j=0}^{n} \left(\sum_{\ell=m_{j-1}+1}^{m_j} f(\gamma(s_\ell)) \left(\gamma(s_\ell) - \gamma(s_{\ell-1})\right) \right|$$

$$\leq \sum_{j=0}^{n} \left(\left| f(\gamma(t_j^*)) \left(\gamma(t_j) - \gamma(t_{j-1})\right) - \sum_{\ell=m_{j-1}+1}^{m_j} f(\gamma(s_\ell)) \left(\gamma(s_\ell) - \gamma(s_{\ell-1})\right) \right| \right.$$

Then,

$$\begin{aligned}
&\left| \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right) \left(\gamma(t_j) - \gamma(t_{j-1})\right) - \sum_{k=1}^{m} f\left(\gamma(s_k^*)\right) \left(\gamma(s_k) - \gamma(s_{k-1})\right) \right| \\
&\qquad = \left| \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right) \left(\gamma(t_j) - \gamma(t_{j-1})\right) - \sum_{j=1}^{n} \left(\sum_{\ell=m_{j-1}+1}^{m_j} f\left(\gamma(s_\ell^*)\right) \left(\gamma(s_\ell) - \gamma(s_{\ell-1})\right) \right) \right| \\
&\qquad \leq \sum_{j=0}^{n} \left(\left| f\left(\gamma(t_j^*)\right) \left(\gamma(t_j) - \gamma(t_{j-1})\right) - \sum_{\ell=m_{j-1}+1}^{m_j} f\left(\gamma(s_\ell^*)\right) \left(\gamma(s_\ell) - \gamma(s_{\ell-1})\right) \right| \right) \\
&\qquad \leq \sum_{j=1}^{n} \left(\sum_{\ell=m_{j-1}+1}^{m_j} \left| f\left(\gamma(t_j^*)\right) - f\left(\gamma(s_\ell^*)\right) \right| \left| \left(\gamma(s_\ell) - \gamma(s_{\ell-1})\right) \right| \right).
\end{aligned}$$

By uniform continuity of $f \circ \gamma$, we have a $\delta > 0$ such that

$$\left| f\left(\gamma(t_j^*)\right) - f\left(\gamma(s_\ell^*)\right) \right| < \frac{\epsilon}{|\gamma|} \qquad\qquad \text{whenever } |t_j^* - s_\ell^*| < \delta.$$

Thus for $|P_1| < \delta$, we have

$$\sum_{j=1}^{n}\left(\sum_{\ell=m_{j-1}+1}^{m_j}\left|f\left(\gamma(t_j^*)\right)-f\left(\gamma(s_\ell^*)\right)\right|\left|\left(\gamma(s_\ell)-\gamma(s_{\ell-1})\right)\right|\right)\leq\frac{\epsilon}{|\gamma|}\sum_{j=1}^{n}\sum_{\ell=m_{j-1}+1}^{m_j}\left|\gamma(s_\ell)-\gamma(s_{\ell-1})\right|$$

$$\leq\frac{\epsilon}{|\gamma|}\sum_{k=1}^{m}\left|\gamma(s_k)-\gamma(s_{k-1})\right|$$

$$\leq\frac{\epsilon}{|\gamma|}|\gamma|=\epsilon.$$

□

Properties

Let $\gamma:[a,b]\longrightarrow\mathbb{C}$ be a rectifiable and $f:\gamma([a,b])\longrightarrow\mathbb{C}$ be continuous.

(1) Let $\tilde{\gamma}:[c,d]\longrightarrow\mathbb{C}$ be a continuous reparametrization of γ. Then

$$\int_\gamma f(z)dz=\int_{\tilde{\gamma}} f(z)dz.$$

PROOF. Let $\varphi:[a,b]\longrightarrow[c,d]$ be a homeomorphism such that $\tilde{\gamma}(\varphi(t))=\gamma(t)$.

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s.t $\tilde{\gamma}(\varphi(t)) = \gamma(t)$.

It is enough to prove that given $\varepsilon>0$, $\exists\ \delta>0$ s.t if $P: a=t_0<t_1<\dots<t_n=b$ is a partition of $[a,b]$ s.t. $|P|<\delta$, then

$$\left|\int_{\tilde{\gamma}} f(z)dz - \sum_{j=0}^{n} f(\gamma(t_j^*))(\gamma(t_j)-\gamma(t_{j-1}))\right| < \varepsilon.$$

It is enough to prove that given $\epsilon>0$, $\exists\delta>0$ such that if $P:a=t_0<t_1<\cdots<t_n=b$ is a partition of $[a,b]$ such that $|P|<\delta$, then

$$\left|\int_{\tilde{\gamma}} f(z)dz-\sum_{j=0}^{n} f\left(\gamma(t_j^*)\right)\left(\gamma(t_j)-\gamma(t_{j-1})\right)\right|<\epsilon.$$

Let $\epsilon > 0$ be given. We know that $\exists\, \delta' > 0$ such that if $Q : c = s_0 < s_1 < \cdots < s_m = b$ such that $Q < \delta'$, then

$$\left| \int_{\tilde{\gamma}} f(z)dz - \sum_{j=0}^{m} f\left(\tilde{\gamma}(s_j^*)\right)\left(\tilde{\gamma}(t_j) - \tilde{\gamma}(t_{j-1})\right) \right| < \epsilon.$$

Since φ is continuous on a compact set, it is uniformly continuous.
Hence $\exists\, \delta > 0$ such that $\left|\varphi(x) - \varphi(y)\right| < \delta'$ whenever $|x - y| < \delta$.

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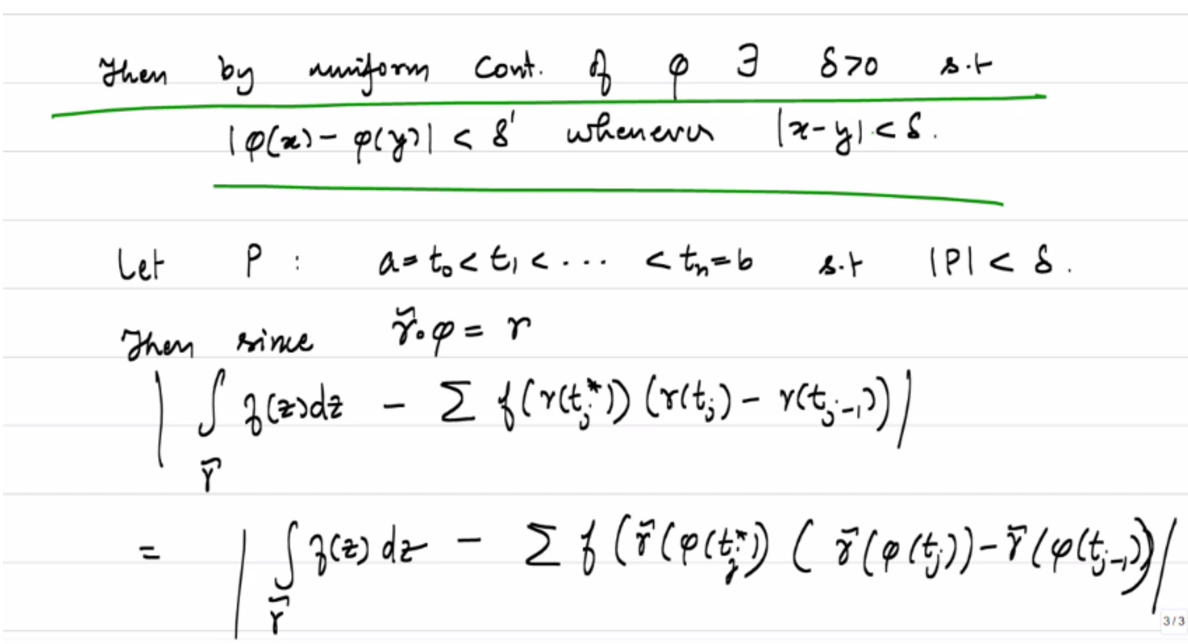

Let $P : a = t_0 < t_1 < \cdots < t_n = b$ such that $|P| < \delta$.
Now it is left as an exercise for the reader to check that φ is monotonically increasing.
Hence, we have $s_0 = \varphi(t_0) < s_1$ Then since $\tilde{\gamma} \circ \varphi = \gamma$,

$$\left|\int_{\tilde{\gamma}} f(z)dz - \sum_{j=0}^{n} f\left(\gamma(t_j^*)\right)\left(\gamma(t_j)-\gamma(t_{j-1})\right)\right|$$
$$= \left|\int_{\tilde{\gamma}} f(z)dz - \sum_{j=0}^{n} f\left(\tilde{\gamma}(\varphi(t_j)^*)\right)\left(\gamma(t_j)-\gamma(t_{j-1})\right)\right|$$

Now,

$$s_0 = \varphi(t_0) < s_1 < \cdots < s_n = \varphi(t_n)$$

where $s_j = \varphi(t_j)$. Moreover $\left|\varphi(t_j) - \varphi(t_{j-1})\right| < \delta'$.

Since $|P| < \delta$,

$$\left|\int_{\tilde{\gamma}} f(z)dz - \sum_{j=0}^{n} f\left(\tilde{\gamma}(\varphi(t_j)^*)\right)\left(\gamma(t_j)-\gamma(t_{j-1})\right)\right|$$
$$= \left|\int_{\tilde{\gamma}} f(z)dz - \sum_{j=0}^{n} f\left(\tilde{\gamma}(s_j^*)\right)\left(\tilde{\gamma}(s_j)-\tilde{\gamma}(s_{j-1})\right)\right| < \epsilon.$$

As ϵ can be arbitrary small, we have

$$\int_{\gamma} f(z)dz = \int_{\tilde{\gamma}} f(z)dz. \qquad \square$$

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2. $\displaystyle\underline{\underline{\int_{-\gamma} f(z)dz}} = -\underline{\underline{\int_{\gamma} f(z)dz}}$.

<u>Proof</u>: Let $\gamma : [a,b] \longrightarrow \mathbb{C}$. Then $(-\gamma): [-b,-a] \longrightarrow \mathbb{C}$ given by $(-\gamma)(t) = \gamma(-t)$.

Let $P : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$. $\Rightarrow$ $P' : -b = s_0 < \cdots < s_n = -a$ where $s_j = -t_{n-j}$.

(2) $\displaystyle\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz.$

PROOF. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$. Then $(-\gamma) : [-b,-a] \longrightarrow \mathbb{C}$ given by $(-\gamma)(t) = \gamma(-t)$. Let $P : a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b] \implies P' : -b = s_0 < s_1 < \cdots < s_n = -a$ where $s_j = -t_{n-j}$.

Given $\epsilon > 0$, let $\delta > 0$ be such that $\displaystyle\int_{(-\gamma)} f(z)dz$ and $\displaystyle\int_{\gamma} f(z)dz$ can be $\epsilon-$ approximated by partition of size less than δ.

Let P be as above with $|P| < \delta$,

$$\begin{aligned}
\left|\int_{-\gamma} f(z)dz + \int_{\gamma} f(z)dz\right| = \Bigg|\int_{-\gamma} f(z)dz - \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right)& \\
+ \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right) + \int_{\gamma} f(z)dz\Bigg|& \\
= \Bigg|\int_{\gamma} f(z)dz - \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right) + \int_{(-\gamma)} f(z)dz& \\
+ \sum_{j=1}^{n} f\left((-\gamma)(s_{n-j}^*)\right)\left((-\gamma)(s_{n-j}) - (-\gamma)(s_{n-j+1})\right)\Bigg|&
\end{aligned}$$

$$\leq \epsilon + \left| \int_{(-\gamma)} f(z)dz - \sum_{j=1}^{n} f\left((-\gamma)(s_j^*)\right)\left((-\gamma)(s_j) - (-\gamma)(s_{j-1})\right) \right|$$

$$< 2\epsilon.$$

Since ϵ is arbitrary, we have

$$\int_{(-\gamma)} f(z)dz = -\int_{\gamma} f(z)dz.$$

$\square$

(3) Let f be defined on $\gamma_1([a,b]) \cup \gamma_2([c,d])$ where γ_1 and γ_2 are rectifiable curves such that $\gamma_1(b) = \gamma_2(c)$. Then

$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

Proof of property (3) is left as an exercise for the reader.

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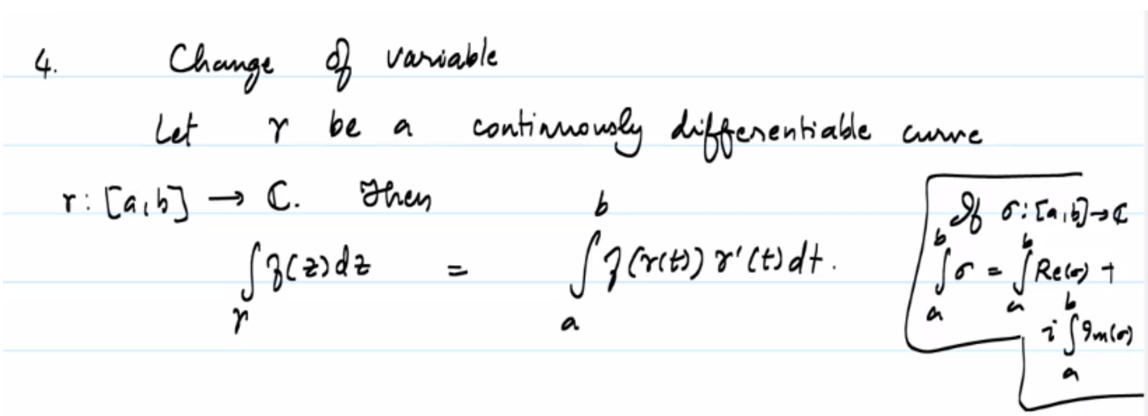

(4) Change of variable.

Let γ be a continuously differentiable curve $\gamma : [a,b] \longrightarrow \mathbb{C}$. Then

$$\int_{\gamma} f(z)dz = \int_a^b f\left(\gamma(t)\right)\gamma'(t)dt.$$

PROOF. Given $\epsilon > 0$, let $\delta > 0$ be such that if $|P| < \delta$,

$$\left|\int_\gamma f(z)dz - \sum_{j=1}^n f\left(\gamma(t_j^*)\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right)\right| < \epsilon$$

for a partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$. Let $\gamma = \varphi + i\psi$. Then,

$$\begin{aligned}\sum_{j=1}^n f\left(\gamma(t_j^*)\right)\left(\gamma(t_j) - \gamma(t_{j-1})\right) &= \sum_{j=1}^n f\left(\gamma(t_j^*)\right)\left(\varphi(t_j) - \varphi(t_{j-1})\right)\\ &\quad + i\sum_{j=1}^n f\left(\gamma(t_j^*)\right)\left(\psi(t_j) - \psi(t_{j-1})\right).\end{aligned}$$

By Mean Value Theorem,

$$\sum_{j=1}^n f\left(\gamma(t_j^*)\right)\left(\varphi(t_j) - \varphi(t_{j-1})\right) = \sum_{j=1}^n f\left(\gamma(t_j^*)\right)\varphi'(s_j^*)(t_j - t_{j-1})$$

where $s_j^* \in [t_{j-1}, t_j]$.

Hence,

$$\begin{aligned}\left|\sum_{j=1}^n f\left(\gamma(t_j^*)\right)\left(\varphi(t_j) - \varphi(t_{j-1})\right)\right| &= \left|\sum_{j=1}^n f\left(\gamma(t_j^*)\right)\left(\varphi'(s_j^*) - \varphi'(t_j^*)\right)(t_j - t_{j-1})\right.\\ &\quad \left. + \sum_{j=1}^n f\left(\gamma(t_j^*)\right)\varphi'(t_j^*)(t_j - t_{j-1})\right|\\ &\le \left|\sum_{j=1}^n f\left(\gamma(t_j^*)\right)\left(\varphi'(s_j^*) - \varphi'(t_j^*)\right)(t_j - t_{j-1})\right|\\ &\quad + \left|\sum_{j=1}^n f\left(\gamma(t_j^*)\right)\varphi'(t_j^*)(t_j - t_{j-1})\right|.\end{aligned}$$

By uniform continuity of φ', let $\delta > 0$ be small enough so that

$$\left|\varphi'(s_j^*) - \varphi'(t_j^*)\right| < \epsilon.$$

If $M = \sup_{[a,b]} f \circ \gamma$ then we have,

$$\left|\sum_{j=1}^n f\left(\gamma(t_j^*)\right)\left(\varphi'(s_j^*) - \varphi'(t_j^*)\right)(t_j - t_{j-1})\right| \le \sum_{j=0}^n M\epsilon(t_j - t_{j-1}) = \epsilon M(b - a).$$

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Therefore as $|P| \to 0$

$$\lim_{|P|\to 0} \sum_{j=0}^{n} f(\gamma(t_j^*))(\varphi(t_j) - \varphi(t_{j-1})) = \lim_{|P|\to 0} \sum f(\gamma(t_j^*))\varphi'(t_j^*)(t_j - t_{j-1})$$

$$= \int_a^b f(\gamma(t))\varphi'(t)\,dt.$$

Therefore, as $|P| \to 0$, $\left|\sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\left(\varphi'(s_j^*) - \varphi'(t_j^*)\right)(t_j - t_{j-1})\right| \to 0.$

Hence,

$$\lim_{|P|\to 0} \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\left(\varphi(t_j) - \varphi(t_{j-1})\right) = \lim_{|P|\to 0} \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\varphi'(t_j^*)(t_j - t_{j-1})$$

$$= \int_a^b f(\gamma(t))\varphi'(t)\,dt.$$

Similarly,

$$\lim_{|P|\to 0} \sum_{j=1}^{n} f\left(\gamma(t_j^*)\right)\left(\psi(t_j) - \psi(t_{j-1})\right) = \int_a^b f(\gamma(t))\psi'(t)\,dt.$$

Hence,

$$\begin{aligned} \int_\gamma f(z)\,dz &= \int_a^b f\left(\gamma(t)\right)\varphi'(t)\,dt + i\int_a^b f\left(\gamma(t)\right)\psi'(t)\,dt \\ &= \int_a^b f\left(\gamma(t)\right)\gamma'(t)\,dt. \end{aligned}$$

□

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Exercises: 5). If $f: \gamma([a,b]) \to \mathbb{C}$ is bounded by M (i.e $\sup_{z \in \gamma([a,b])} |f(z)| = M$). Then

$$\int_\gamma f(z)dz \leq M|\gamma|.$$

(5) If $f : \gamma([a,b]) \longrightarrow \mathbb{C}$ is bounded by M, i.e., $\sup_{z\in\gamma([a,b])} |f(z)| = M$. Then

$$\int_\gamma f(z)dz \leq M|\gamma|.$$

(6) Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ and f, g be defined on $\gamma([a,b])$. Then for $c \in \mathbb{C}$,

$$\int_\gamma (cf+g)(z)dz = c\int_\gamma f(z)dz + \int_\gamma g(z)dz.$$

EXERCISE 2. Prove property (5) and property (6).