Complex Analysis

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Module No - 3

Lecture No – 16

Curves in the Complex Plane

In the last few weeks we explore the notion of differentiability in great detail. Just like in the real analysis setting, the notion of complex differentiability is also tied down together with notion of integration. And among the notions of integration, we will be most interested in integral along curves or rather line integrals. We will plunge into that in the next few weeks but in this lecture let us review and recall the notion of curve and the various properties of curves. Let us begin this lecture by recalling what are curves?

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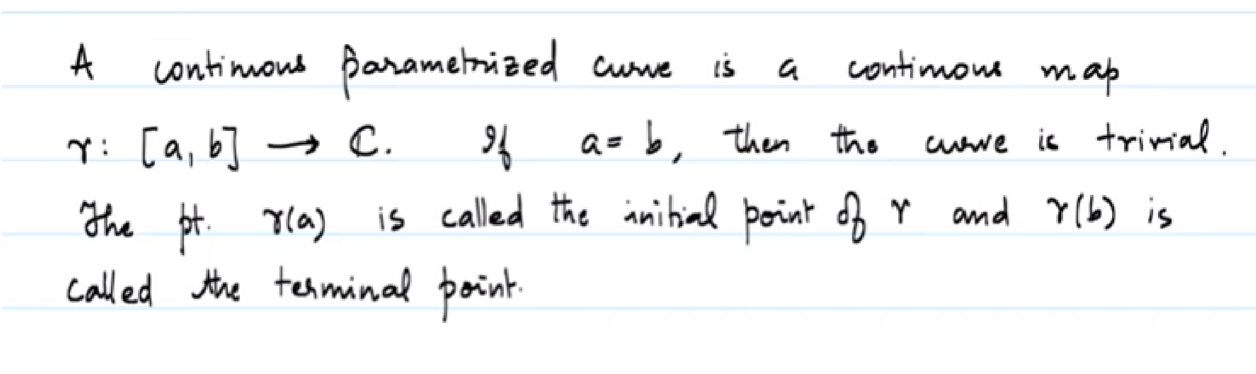

DEFINITION 1. A continuous parametrized curve is a continuous map $\gamma : [a, b] \longrightarrow \mathbb{C}$.

If $a = b$, then the curve is trivial. The point $\gamma(a)$ is called the initial point of γ and $\gamma(b)$ is called the terminal point.

γ is said to be a closed curve if $\gamma(a) = \gamma(b)$. We say that γ is a simple curve if $\gamma(t) \neq \gamma(t')$ with the exception of $t = a, t' = b$.

The image $\gamma([a, b])$ of γ is called the image of the curve.

EXAMPLE 1.

- Let $z_1, z_2 \in \mathbb{C}$,

$$\gamma_{z_1 \to z_2} := (1-t)z_1 + tz_2, \qquad t \in [0,1].$$

  If $z_1 = z_2$, then $\gamma_{z_1 \to z_2}$ is a closed curve but not simple.
- Define $\gamma(\theta) = z_0 + re^{i\theta},\ \theta \in [0, 2\pi]$. Then image of curve is a circle of radius r centered z_0. Here the initial and terminal points is $z_0 + r$. Hence γ is a simple closed curve.

  Now consider γ_1 and γ_2 defined by

$$\gamma_1(\theta) = z_0 + re^{2\pi i\theta},\ \theta \in [0,1]$$

$$\gamma_2(\theta) = z_0 + re^{2i\theta}\ \theta \in [0, 2\pi].$$

  Then γ_1 is a simple closed curve having initial and terminal point $z_0 + r$ and image of γ_1 is same as that of γ, but still we treat them as different curves as domain of γ and γ_1 are different. In the case of γ_2, it looks similar to γ, however γ_2 is not a simple closed curve as for any point in the image of γ_2 we have two preimages.

Now we would like to somehow identify γ and γ_1, and the right notion to look at for this purpose of curves is the continuous reparametrization.

(Refer Slide Time: 10:21)

We say that a curve $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ is a continuous reparametrization of $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ if $\exists$ a homeomorphism $\varphi : [a_1, b_1] \to [a_2, b_2]$ s.t. $\varphi(a_1) = a_2$ & $\varphi(b_1) = b_2$ and s.t

$$\gamma_2(\varphi(t)) = \gamma_1(t) \quad \forall \; t \in [a_1, b_1].$$

$\gamma_{z_2 \to z_1}$ is not a reparametrization of $\gamma_{z_1 \to z_1}$

DEFINITION 2. We say that a curve $\gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$ is a continuous reparametrization of $\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C}$ if $\exists$ a homeomorphism $\varphi : [a_1, b_1] \longrightarrow [a_2, b_2]$ such that $\varphi(a_1) = a_2$ and $\varphi(b_1) = b_2$ and such that $\gamma_2(\varphi(t)) = \gamma_1(t), \; \forall \, t \in [a_1, b_1]$.

By the definition, reparametrized curves must have the same initial and terminal point and also the image must be same. Hence $\gamma_{z_2 \to z_1}$ is not a reparametrization of $\gamma_{z_1 \to z_2}$

EXERCISE 2. Continuous reparametrization is an equivalence relation.

DEFINITION 3. We say that a curve $-\gamma$ is a reversal of a curve $\gamma : [a, b] \to \mathbb{C}$ if $-\gamma : [-b, -a] \to \mathbb{C}$ and $-\gamma(t) = \gamma(-t)$

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$$-\gamma(t) = \gamma(-t).$$

Example: ✓ $\underbrace{\gamma_{z_2 \to z_1}}_{[0,1]} \equiv \underbrace{-\gamma_{z_1 \to z_2}}_{[-1,0]}$.

EXAMPLE 3. Let $\gamma_{z_2\to z_1}$ be the straight line joining z_2 to z_1, then

$$\gamma_{z_2\to z_1} \equiv -\gamma_{z_1\to z_2}$$

where $\gamma_{z_1\to z_2}$is a straight line joining z_1 to z_2.

Now let us define the notion of concatenation.

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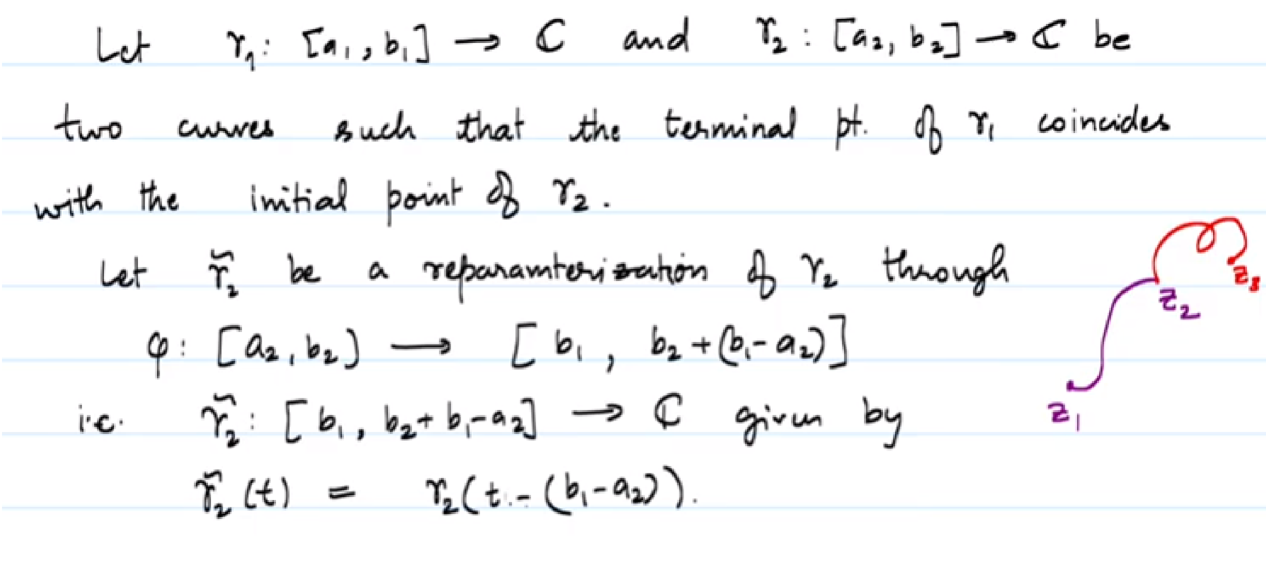

Let $\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$ be two curves such that the terminal point of γ_1 coincides with the initial point of γ_2. Let $\tilde{\gamma}_2$ be a reparametrization of γ_2 through $\varphi : [a_2, b_2] \longrightarrow [b_1, b_2 + (b_1 - a_2)]$. That is, $\tilde{\gamma}_2 : [b_1, b_2 + (b_1 - a_2)] \longrightarrow \mathbb{C}$ given by

$$\tilde{\gamma}_2(t) = \gamma_2(t - (b_1 - a_2)).$$

Define

$$\gamma_1 + \gamma_2 : [a_1, b_2 + b_1 - a_2] \longrightarrow \mathbb{C}$$

by

$$(\gamma_1+\gamma_2)(t) := \begin{cases} \gamma_1(t), & a_1 \le t \le b_1 \\ \tilde{\gamma}_2(t), & b_1 \le t \le b_2 + b_1 - a_2. \end{cases}$$

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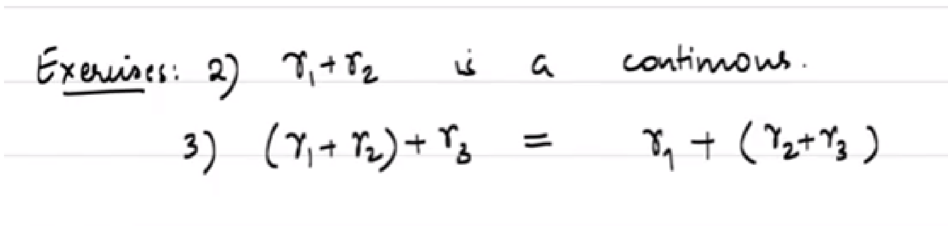

EXERCISE 4.

- $\gamma_1+\gamma_2$ is continuous.
- Let $\gamma_1, \gamma_2, \gamma_3$ be three curves such that terminal point of γ_1 is the initial point of γ_2 and terminal point of γ_2 is the initial point of γ_3. Then,

$$(\gamma_1+\gamma_2)+\gamma_3 = \gamma_1+(\gamma_2+\gamma_3).$$

PROPOSITION 5. *Let $\gamma_1, \gamma_2, \tilde{\gamma}_1, \tilde{\gamma}_2$ be curves such that terminal point of γ_1 coincides with the initial point of γ_2. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be continuous reparametrization of γ_1 and γ_2 respectively. Then,*

$$\gamma_1+\gamma_2 \equiv \tilde{\gamma}_1+\tilde{\gamma}_2$$

and

$$-\gamma_1 = -\tilde{\gamma}_1.$$

PROOF. Let $\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C}$ and $\tilde{\gamma}_1 : [c_1, d_1] \longrightarrow \mathbb{C}$. Let $\varphi_1 : [a_1, b_1] \longrightarrow [c_1, d_1]$ be the homeomorphism such that $\varphi_1(a_1) = c_1$ and $\varphi_1(b_1) = d_1$ and such that $\tilde{\gamma}_1(\varphi_1(t)) = \gamma_1(t),\ \forall\, t \in [a_1, b_1]$. Similarly, let $\gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$ and $\tilde{\gamma}_2 : [c_2, d_2] \longrightarrow \mathbb{C}$. Let $\varphi_2 : [a_2, b_2] \longrightarrow [c_2, d_2]$ be homeomorphism.

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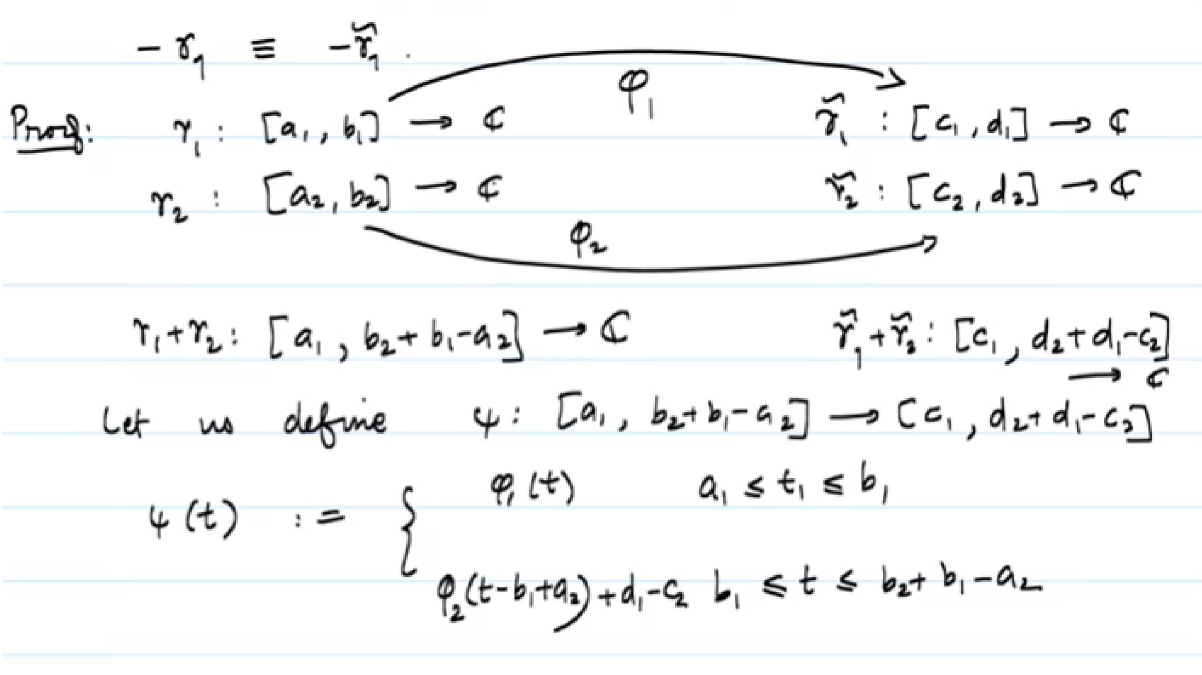

Now $\gamma_1 + \gamma_2 : [a_1, b_2 + b_1 - a_2] \longrightarrow \mathbb{C}$ and $\tilde{\gamma}_1 + \tilde{\gamma}_2 : [c_1, d_2 + d_1 - c_2] \longrightarrow \mathbb{C}$.

Let us define $\psi : [a_1, b_2 + b_1 - a_2] \longrightarrow [c_1, d_2 + d_1 - c_2]$ by

$$\psi(t) := \begin{cases} \varphi_1(t), & a_1 \le t \le b_1 \\ \varphi_2(t - b_1 + a_2) + d_1 - c_2, & b_1 \le t \le b_2 + b_1 - a_2. \end{cases}$$

Check that ψ is a homeomorphism. Hence $\gamma_1 + \gamma_2 \equiv \tilde{\gamma}_1 + \tilde{\gamma}_2$.

Now to prove other result,

$-\gamma_1 : [-b_1, -a_1] \longrightarrow \mathbb{C}$ and $-\tilde{\gamma}_1 : [-d_1, -c_1] \longrightarrow \mathbb{C}$. Since $-\varphi : [-b_1, -a_1] \longrightarrow [-d_1, -c_1]$ and $-\varphi$ is a homeomorphism (Why?) we have the result. $\square$

EXERCISE 6. $-(\gamma_1 + \gamma_2) \equiv (-\gamma_1) + (-\gamma_2)$.

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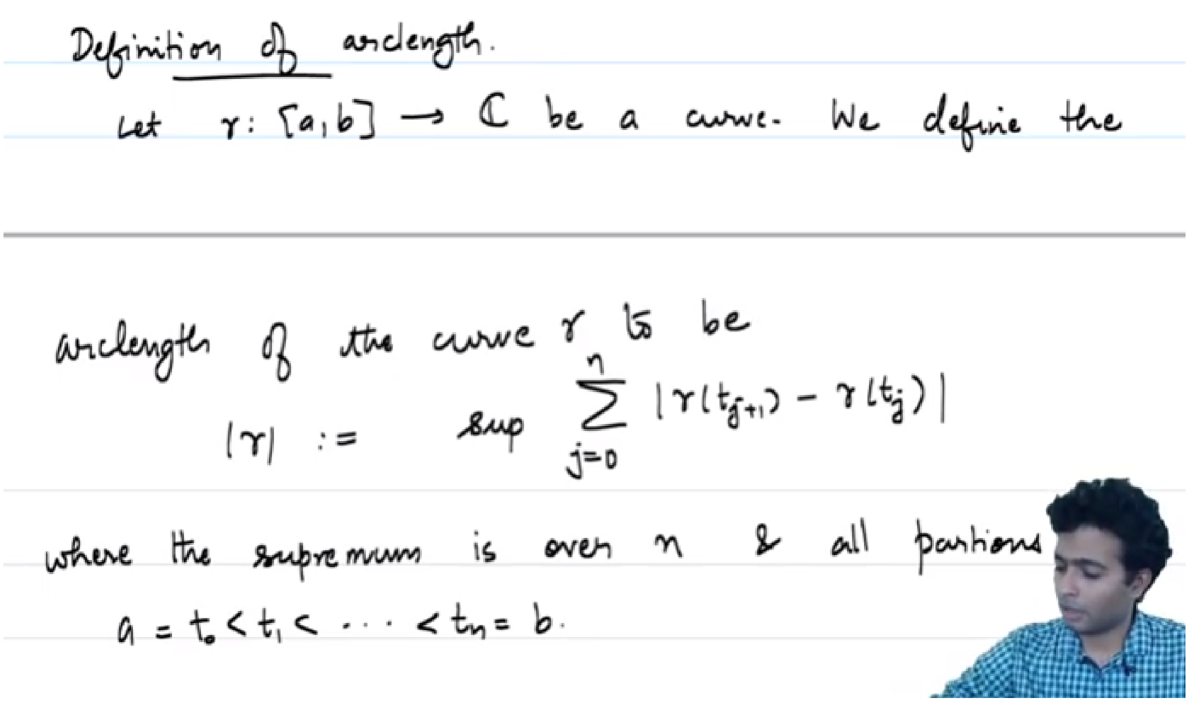

Definition 4 (Arc-length). Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a curve. We define the arc-length of the curve γ to be

$$|\gamma| := \sup \sum_{j=0}^{n} |\gamma(t_j) - \gamma(t_{j-1})|$$

where the supremum is over n and all partitions $a = t_0 < t_1 < \cdots < t_n = b$.

We say that the curve γ is rectifiable if $|\gamma|$ is finite.

Exercise 7. $|\gamma_1 + \gamma_2| = |\gamma_1| + |\gamma_2|$.

Definition 5. A curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ is said to be continuously differentiable if for each $t_0 \in (a,b)$,

$$\gamma'(t_0) = \lim_{\substack{t \to t_0 \\ t \in [a,b] \setminus \{t_0\}}} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists and is continuous.

(Refer Slide Time: 37:35)

<u>Lemma</u>: Let $g : [a,b] \to \mathbb{C}$ be cont. Then

$$\left| \int_a^b g(t)\, dt \right| \leqslant \int_a^b | g(t) |\, dt.$$

$$\left[\int_a^b g(t)\, dt \;:=\; \int_a^b \mathrm{Re}\, g(t) dt + i \int_a^b \mathrm{Im}(g(t))\, dt \right].$$

LEMMA 8. *Let $g : [a,b] \longrightarrow \mathbb{C}$ be continuous. Then*

$$\left| \int_a^b g(t)dt \right| \le \int_a^b |g(t)| dt.$$

Here, $\int_a^b g(t)dt := \int_a^b \mathfrak{Re}(g(t))dt + i \int_a^b \mathfrak{Im}(g(t))dt$.

PROOF. Fix $\theta \in \mathbb{R}$,

$$\begin{aligned}
\left| \mathfrak{Re}\left(e^{i\theta} \int_a^b g(t)dt \right) \right| &= \left| \int_a^b \mathfrak{Re}\left(e^{i\theta} g(t) \right) dt \right| \\
&\le \int_a^b \left| \mathfrak{Re}\left(e^{i\theta} g(t) \right) \right| dt \\
&\le \int_a^b \left| e^{i\theta} g(t) \right| dt \\
&= \int_a^b |g(t)| dt.
\end{aligned}$$

Taking supremum, we get

$$\left| \int_a^b g(t)dt \right| \le \int_a^b |g(t)| dt.$$

□

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Theorem: Let $\gamma : [a,b] \to \mathbb{C}$ be a continuously diff. curve Then γ is rectifiable and

$$|\gamma| = \int_a^b |\gamma'(t)| dt. \quad \text{(Arc-length formula).}$$

THEOREM 9. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a continuous differentiable curve. Then γ is rectifiable and*

$$|\gamma| = \int_a^b |\gamma'(t)| dt.$$

The formula $|\gamma| = \int_a^b |\gamma'(t)| dt$ is called the arc-length formula.

PROOF. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$. Using the fundamental theorem of calculus,

$$\begin{aligned}
\sum_{j=0}^{n} \left|\gamma(t_j) - \gamma(t_{j-1})\right| &= \sum_{j=0}^{n} \left| \int_{t_{j-1}}^{t_j} \gamma'(t) dt \right| \\
&\le \sum_{j=0}^{n} \int_{t_{j-1}}^{t_j} \left|\gamma'(t)\right| dt \\
&= \int_a^b |\gamma'(t)| dt.
\end{aligned}$$

Hence taking supremum over partitions, we get

$$|\gamma| \le \int_a^b |\gamma'(t)| dt.$$

From this, we get $|\gamma|$ is finite and hence a continuous differentiable curve is rectifiable.

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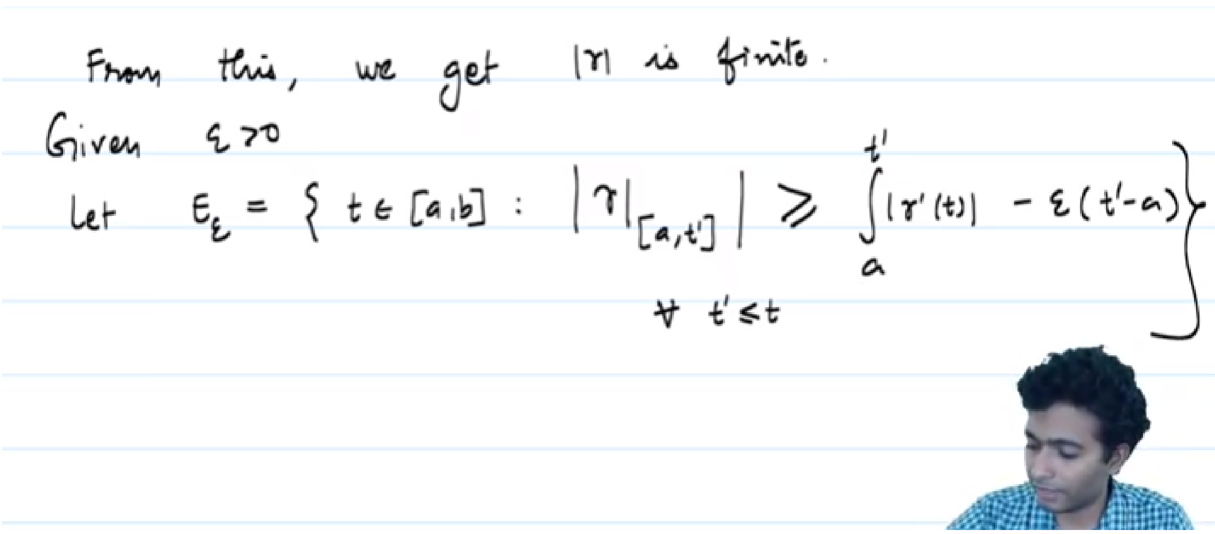

To prove the other inequality, consider the following set.

Given $\epsilon > 0$, let

$$E_\epsilon = \left\{ t \in [a,b] : \left| \gamma \restriction_{[a,t']} \right| \geq \int_a^{t'} |\gamma'(t)| dt - \epsilon(t'-a), \forall\, t' \leq t \right\}.$$

Claim: E_ϵ is closed.

Let $t_n \in E_\epsilon$ be a sequence converging to t_0. We want to show that

$$\left| \gamma \restriction_{[a,t']} \right| \geq \int_a^{t'} |\gamma'(t) dt| - \epsilon(t'-a), \forall\, t' \leq t_0.$$

For, $t' < t_0$, this is satisfied. Because t_n is a sequence which converges to t_0. So, if $t' < t_0$ is fixed then there will be t_m for some m which will be greater than t'. Since $t_n \in E_\epsilon$ and $t' < t_m \implies t'$ satisfies the condition.

Now let's check the condition for $t' = t_0$. If $\{t_n\}_{n\in\mathbb{N}}$ is a sequence converging to t_0 from the right, again we are done by above argument.

Let us assume that $\{t_n\}_{n\in\mathbb{N}}$ is a sequence converging from the left to t_0. Then we have following observation,

$$\left| \gamma \restriction_{[a,t_0]} \right| \geq \left| \gamma \restriction_{[a,t_n]} \right| \geq \int_a^{t_n} |\gamma'(t)| dt - \epsilon(t_n - a).$$

Hence $t_0 \in E_\epsilon$.

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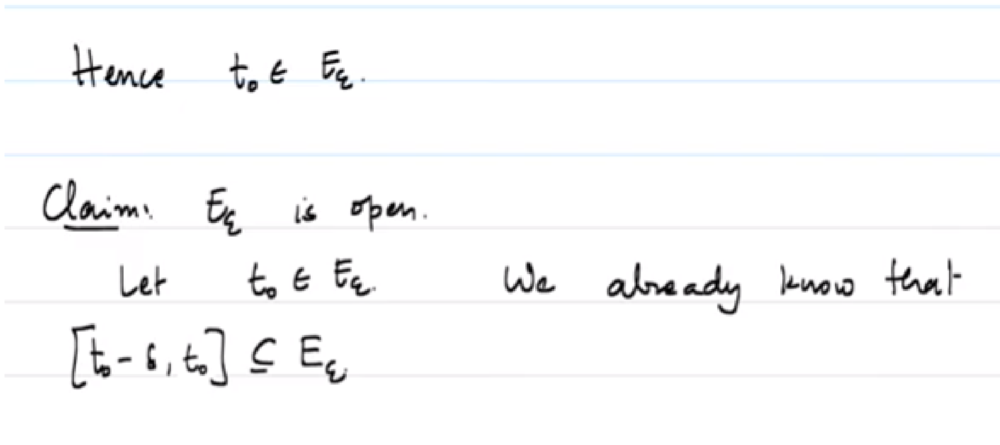

Claim: E_ϵ is open.

Let $t_0 \in E_\epsilon$. We already know that $[t_0 - \delta, t_0] \subset E_\epsilon$ for δ small enough.

Since γ is continuously differentiable at t_0, given $\epsilon > 0$, $\exists \delta > 0$ such that

$$\left| \frac{\gamma(t) - \gamma(t')}{t - t'} - \gamma'(t_0) \right| \leq \frac{\epsilon}{2}, \qquad \text{whenever } (t - t_0) < \delta.$$

Then,

$$|\gamma'(t_0)(t - t_0)| - |\gamma(t) - \gamma(t_0)| \leq \frac{\epsilon}{2}(t - t_0).$$

$$|\gamma(t) - \gamma(t_0)| \geq |\gamma'(t_0)|(t - t_0) - \frac{\epsilon}{2}(t - t_0). \to (*)$$

Since γ' is continuous, we can pick $\delta > 0$ small enough so that for $t \in (t_0, t + \delta)$,

$$\int_{t_0}^{t} |\gamma'(t)| dt \leq \int_{t_0}^{t} \left(|\gamma'(t_0)| + \frac{\epsilon}{2} \right) dt \leq |\gamma'(t_0)|(t - t_0) + \frac{\epsilon}{2}(t - t_0). \to (**)$$

Using $(*)$ and $(**)$,

$$\int_{t_0}^{t} |\gamma'(t)| dt \leq |\gamma(t) - \gamma(t_0)| + \frac{\epsilon}{2}(t - t_0).$$

Now,

$$\left| \gamma \restriction_{[t_0, t]} \right| \geq |\gamma(t) - \gamma(t_0)| \geq \int_{t_0}^{t} |\gamma'(t)| dt - \frac{\epsilon}{2}(t - t_0).$$

Since $t_0 \in E_\epsilon$,

$$\left| \gamma \restriction_{[a, t_0]} \right| \geq \int_{a}^{t_0} |\gamma'(t)| dt - \frac{\epsilon}{2}(t_0 - a).$$

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$$\left| \gamma|_{[a,t]} \right| \geqslant \int_a^t |\gamma'(t)|\,dt - \varepsilon(t-a).$$

$$\forall \; t \in [a, t_0+\delta)$$

$\Rightarrow$ E_ε is open.

$\Rightarrow$ $E_\varepsilon = [a,b]$

Hence,

$$\left|\gamma\restriction_{[a,t]}\right| \geq \int_a^t |\gamma'(t)|dt - \epsilon(t-a), \forall\, t \in [a, t_0+\delta).$$

Then, E_ϵ is open and $[a,b]$ is connected $\implies E_\epsilon = [a,b]$.

Hence $\forall\, \epsilon > 0$,

$$|\gamma| \geq \int_a^b |\gamma'(t)|dt - \epsilon(b-a).$$

Hence, $|\gamma| \geq \displaystyle\int_a^b |\gamma'(t)|dt$.

Therefore $|\gamma| = \displaystyle\int_a^b |\gamma'(t)|dt$. $\square$

EXERCISE 10.

- $\left|\gamma_{z_1 \to z_2}\right| = |z_2 - z_1|$.
- Let $\gamma(\theta) = z_0 + re^{i\theta}$ for $\theta \in [0, 2\pi]$. Then $|\gamma| = 2\pi r$.