Complex Analysis

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Module No - 3

Lecture No – 15

Problem Session

In this problem session, we will address some problems on the Cauchy- Riemann equations and its implications. Recall that given $\Omega \subseteq \mathbb{C}$ a domain and $f : \Omega \to \mathbb{C}$ which is differentiable. Then the Wirtinger derivatives were defined as

$$\frac{\partial f}{\partial z}(z_0) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right)$$

$$\frac{\partial f}{\partial \bar{z}}(z_0) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right).$$

PROBLEM 1. Let f be a polynomial given by

$$f(z) = \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} z^n \bar{z}^m.$$

Show that

$$\frac{\partial f}{\partial z}(z) = \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m}(nz^{n-1})\bar{z}^m$$

and

$$\frac{\partial f}{\partial \bar{z}}(z) = \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} z^n (m\bar{z}^{m-1}).$$

(Refer Slide Time: 04:38)

Proof: Claim: $\frac{\partial}{\partial z}(f+g) = \frac{1}{2}\left(\frac{\partial}{\partial x}(f+g) + \frac{1}{i}\frac{\partial}{\partial y}(f+g)\right)$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial g}{\partial x} + \frac{1}{i}\frac{\partial g}{\partial y}\right)$$

$$= \partial f/\partial z + \partial g/\partial z.$$

SOLUTION 1. Before going to the solution of the problem, we first observe some of the properties satisfied by the Wirtinger derivatives.

$$\begin{aligned}\frac{\partial}{\partial z}(f+g) &= \frac{1}{2}\left(\frac{\partial}{\partial x}(f+g) + \frac{1}{i}\frac{\partial}{\partial y}(f+g)\right)\\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial g}{\partial x} + \frac{1}{i}\frac{\partial g}{\partial y}\right)\\ &= \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}.\end{aligned}$$

Similarly

$$\frac{\partial}{\partial \bar{z}}(f+g) = \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}.$$

Wirtinger derivatives satisfies the product rule also:

$$\begin{aligned}\frac{\partial}{\partial z}(fg) &= \frac{1}{2}\left(\frac{\partial}{\partial x}(fg) + \frac{1}{i}\frac{\partial}{\partial y}(fg)\right)\\ &= \frac{1}{2}\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x} + \frac{1}{i}\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\right)\\ &= f\left(\frac{1}{2}\left(\frac{\partial g}{\partial x} + \frac{1}{i}\frac{\partial g}{\partial y}\right)\right) + g\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)\right)\\ &= f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}.\end{aligned}$$

Similarly

$$\frac{\partial}{\partial \bar{z}}(fg) = f\frac{\partial g}{\partial \bar{z}} + g\frac{\partial f}{\partial \bar{z}}.$$

Now observe that

$$\frac{\partial \bar{z}}{\partial z} = \frac{1}{2}\left(\frac{\partial \bar{z}}{\partial x} + \frac{1}{i}\frac{\partial \bar{z}}{\partial y}\right) = \frac{1}{2}(1-1) = 0.$$

Hence by applying product rule, we have $\dfrac{\partial \bar{z}^m}{\partial z} = 0$.

We also know that

$$\frac{\partial z^n}{\partial z} = nz^{n-1}.$$

Let $f(z) = \sum\limits_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} z^n \bar{z}^m$.

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Hence $\dfrac{\partial f}{\partial z} = \sum\limits_{\substack{n,m\geq 0\\ n+m\leq d}} C_{n,m}\,(n z^{n-1})\,\bar{z}^m$.

In order to prove the second expression $\dfrac{\partial \bar{z}}{\partial \bar{z}} = 0$.

& $\dfrac{\partial \bar{z}^m}{\partial \bar{z}} = m\bar{z}^{m-1}$.

$$\begin{aligned}
\frac{\partial f}{\partial z}(z) &= \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} \frac{\partial}{\partial z}\left(z^n \bar{z}^m\right) \\
&= \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} \left(\bar{z}^m \frac{\partial z^n}{\partial z} + z^n \frac{\partial \bar{z}^m}{\partial z}\right) \\
&= \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} (nz^{n-1})\bar{z}^m.
\end{aligned}$$

In order to prove the second expression, observe that $\frac{\partial z}{\partial \bar{z}} = 0$ and hence $\frac{\partial z^n}{\partial \bar{z}} = 0$. Also $\frac{\partial z^{\bar{m}}}{\partial \bar{z}} = m\bar{z}^{m-1}$.Now it is an easy exercise for the reader to complete the solution.

Before we do the next problem, let us also look at the Laplacian operator in terms of the Wirtinger derivatives.

Recall that the Laplacian operator was given by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Now by the definition of Wirtinger derivatives, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}$$

and

$$i\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}.$$

(Refer Slide Time: 15:44)

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\right) + \frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\right)$$

$$= \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial \bar{z}^2} + 2\frac{\partial^2 f}{\partial z \partial \bar{z}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{i}\left(\frac{\partial}{\partial z}\left(\frac{1}{i}\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)\right) - \frac{\partial}{\partial \bar{z}}\left(\frac{1}{i}\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)\right)\right).$$

Hence

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \bar{z}}\right)+\frac{\partial}{\partial \bar{z}}\left(\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \bar{z}}\right)\\ &= \frac{\partial^2 f}{\partial z^2}+\frac{\partial^2 f}{\partial \bar{z}^2}+2\frac{\partial^2 f}{\partial z\partial \bar{z}}\\ \frac{\partial^2 f}{\partial y^2} &= \frac{1}{i}\left(\frac{\partial}{\partial z}\left(\frac{1}{i}\left(\frac{\partial f}{\partial z}-\frac{\partial f}{\partial \bar{z}}\right)\right)-\frac{\partial}{\partial \bar{z}}\left(\frac{1}{i}\left(\frac{\partial f}{\partial z}-\frac{\partial f}{\partial \bar{z}}\right)\right)\right)\\ &= -\left(\frac{\partial^2 f}{\partial z^2}+\frac{\partial^2 f}{\partial \bar{z}^2}-2\frac{\partial^2 f}{\partial z\partial \bar{z}}\right).\end{aligned}$$

Hence

$$\Delta f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} = 4\frac{\partial^2 f}{\partial z\partial \bar{z}}.$$

From the previous problem we can see that a polynomial $f(z) = \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} z^n \bar{z}^m$ is holomorphic if and only if $c_{n,m} = 0$ whenever $m > 0$. Just like how we can give a characterization of what holomorphic polynomials will look like, we will also be able to talk about how harmonic polynomials will be look like.

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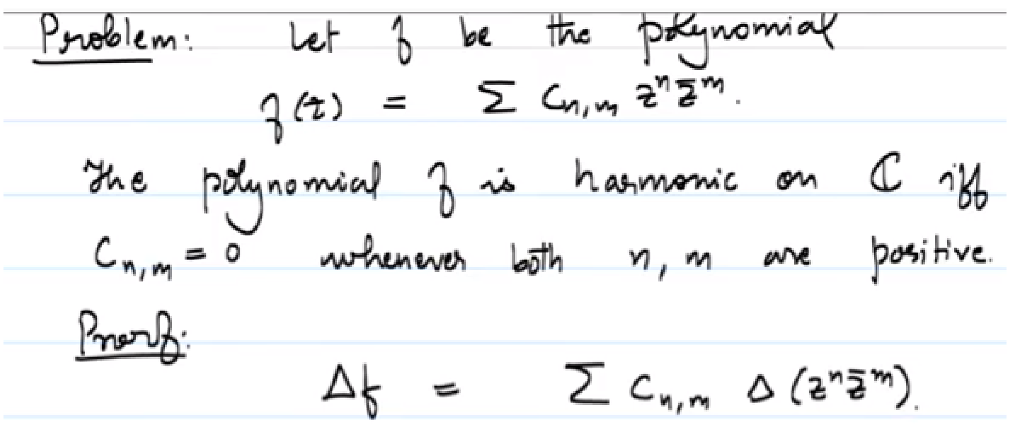

PROBLEM 2. Let f be the polynomial

$$f(z) = \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} z^n \bar{z}^m.$$

Then the polynomial f is harmonic on $\mathbb{C}$ if and only if $c_{n,m} = 0$ whenever both n, m are positive.

SOLUTION 2.

$$\Delta f(z) = \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m}\Delta\left(z^n\bar{z}^m\right).$$

Now,

$$\Delta(z^n) = 4\frac{\partial^2(z^n)}{\partial z\partial\bar{z}} = 4\frac{\partial}{\partial z}\left(\frac{\partial z^n}{\partial\bar{z}}\right) = 0$$

and

$$\Delta(\bar{z}^m) = 4\frac{\partial^2(\bar{z}^m)}{\partial\bar{z}\partial z} = 4\frac{\partial}{\partial\partial\bar{z}}\left(\frac{\partial\bar{z}^m}{\partial z}\right) = 0.$$

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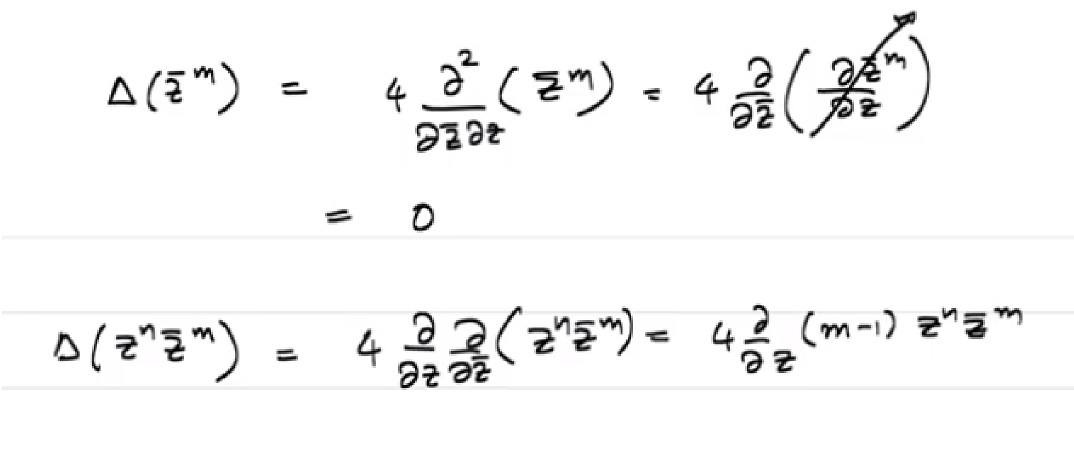

By the properties of Wirtinger derivatives we saw,

$$\begin{aligned}\Delta\left(z^n\bar{z}^m\right) &= 4\frac{\partial^2}{\partial z\partial\bar{z}}\left(z^n\bar{z}^m\right)\\ &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}}\left(z^n\bar{z}^m\right)\\ &= 4\frac{\partial}{\partial z}\left((m-1)z^n\bar{z}^{m-1}\right)\\ &= 4(m-1)(n-1)z^{n-1}\bar{z}^{m-1}.\end{aligned}$$

(Refer Slide Time: 24:02)

$$= 4(n-1)(m-1)z^{n-1}\bar{z}^{m-1}$$

$$\Delta f(z) = \sum_{\substack{n,m>0\\ n+m\leq d}} c_{n,m}\, 4(n-1)(m-1)\underbrace{z^{n-1}\bar{z}^{m-1}}$$

$$\Delta f(z) = 0 \quad \text{iff} \quad c_{n,m} = 0 \quad \forall \; n,m>0.$$

$$\text{—} \blacksquare$$

Therefore,

$$\Delta f(z) = \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} 4(m-1)(n-1)z^{n-1}\bar{z}^{m-1}.$$

Hence $\Delta f(z) = 0$ if and only if $c_{n,m} = 0 \;\forall\; n,m>0$.

PROBLEM 3. Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function. Then f is real valued only if f is constant.

SOLUTION 3. Let $f = u + iv$ be real valued $\implies v = 0$.

(Refer Slide Time: 27:34)

$$u(x+iy) = u(0) + \int_0^x \frac{\partial u}{\partial x}(t)\,dt + \int_0^y \frac{\partial u}{\partial y}(x+it)\,dt$$

$$= u(0) + \int_0^x \cancel{\frac{\partial v}{\partial y}}^{\,0}(t)\,dt - \int_0^y \cancel{\frac{\partial v}{\partial x}}^{\,0}(x+it)\,dt.$$

$$\implies \quad u(z) = u(0) \qquad \text{—} \blacksquare$$

Then,

$$\begin{aligned}
u(x+iy) &= u(0) + \int_0^x \frac{\partial u}{\partial x}(t)\,dt + \int_0^y \frac{\partial u}{\partial y}(x+it)\,dt \\
&= u(0) + \int_0^x \frac{\partial v}{\partial y}(t)\,dt - \int_0^y \frac{\partial v}{\partial x}(x+it)\,dt \\
&= u(0).
\end{aligned}$$

Hence $u(z) = u(0)$.