Complex Analysis

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Module No - 3

Lecture No – 14

Möbius transformations

In this lecture, we will discuss a very important class of function called Möbius transformation. Möbius transformations are special rational function which naturally occurs in study of various domains of complex analysis. But before we get into the study of Möbius transformations, it will be useful to study the extended complex plane. The extended complex plane is nothing but the complex plane with 1 point attached to it.

If you have seen course on topology it is just a 1 point compactification of the complex plane. Let me begin by what an extended complex plane is and we will give a concrete description of what that is.

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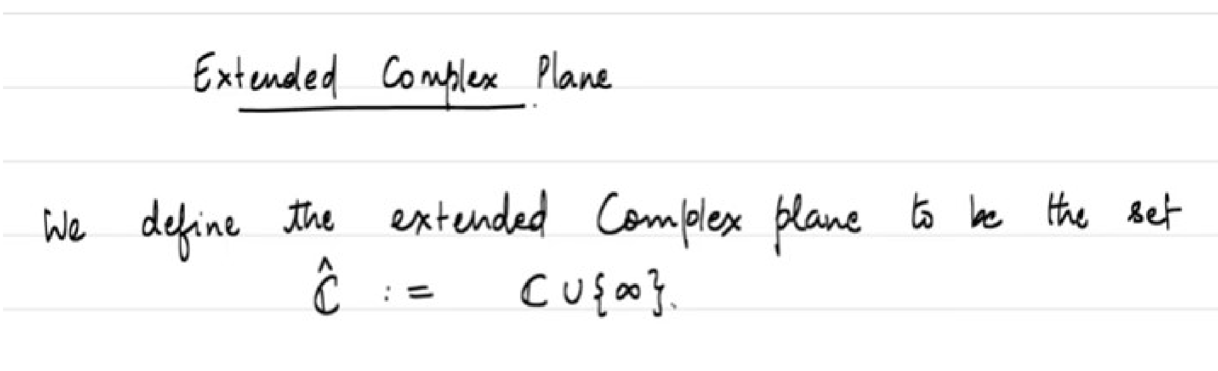

<u>**Extended Complex Plane**</u>

We define extended complex plane to be the set $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

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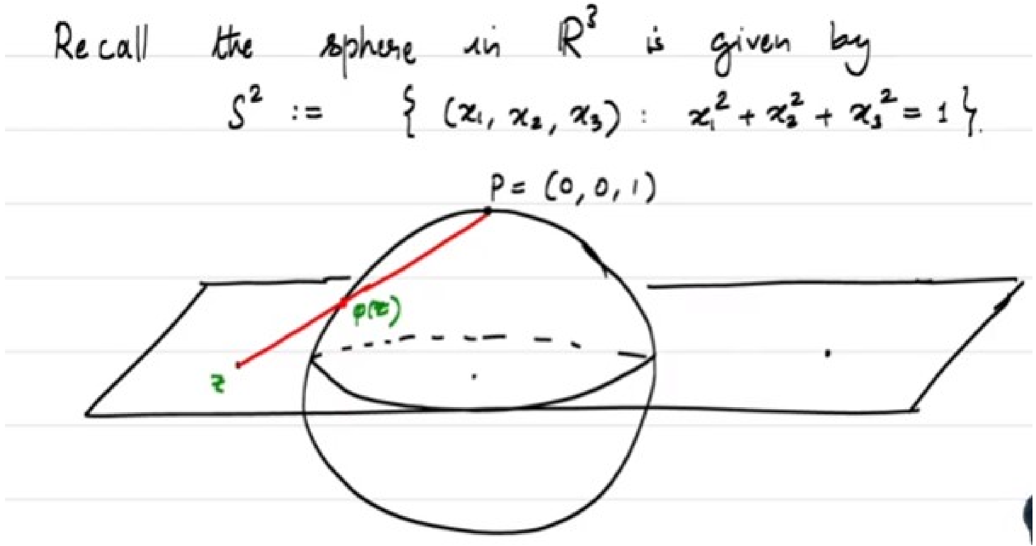

Recall that the sphere in $\mathbb{R}^3$ is given by $S^2 := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$.

Let $P = (0,0,1)$ and let $z = (x, y, 0)$ be a point in $\mathbb{C}$, then the line joining z and P is given by $L = \{(1-t)z + tP : t \in \mathbb{R}\}$. That is $L = \{\big((1-t)x, (1-t)y, tz)\big) : t \in \mathbb{R}\}$.

Now let us find where this line meets the unit sphere.

$$(1-t)^2 x^2 + (1-t)^2 y^2 + t^2 = 1$$

$$(1-t)^2 |z|^2 = (1-t^2)$$

For $t \neq 1$,

$$|z|^2 = \frac{(1+t)}{(1-t)} \implies t = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Hence L and S^2 meets at $\left(\dfrac{2x}{|z|^2+1}, \dfrac{2y}{|z|^2+1}, \dfrac{|z|^2-1}{|z|^2+1}\right)$.

Define $\varphi : \mathbb{C} \longrightarrow S^2$ given by

$$\varphi(z) = \left(\frac{2\mathfrak{Re}(z)}{|z|^2+1}, \frac{2\mathfrak{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

By defining $\varphi(\infty) = P$, we have a bijection from $\hat{\mathbb{C}}$ to S^2. The map φ defined is called the steriographic projection.

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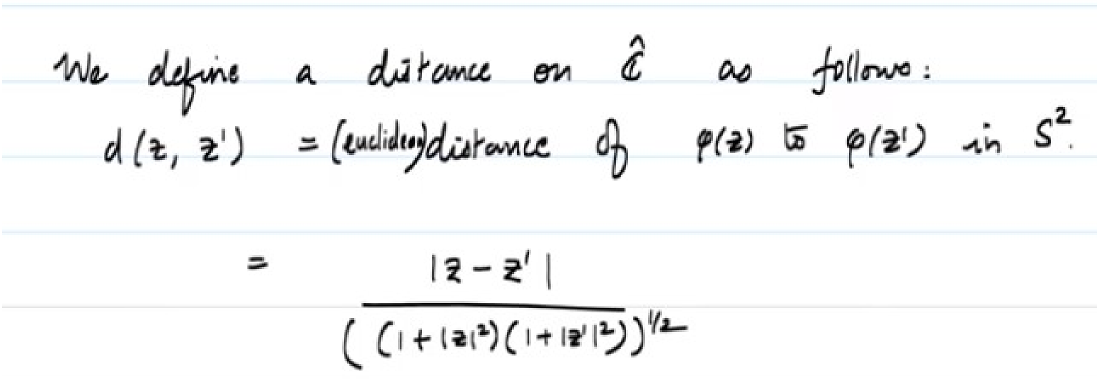

We define a distance on $\hat{\mathbb{C}}$ as follows:

$$d(z, z') = \text{Euclidean distance of } \varphi(z) \text{ to } \varphi(z') \text{ in } S^2$$
$$= \frac{2|z - z'|}{\big((1+|z|^2)(1+|z'|^2)\big)^{1/2}}.$$

If $z' = \infty$, then

$$d(z, z') = \frac{2}{1+|z|^2}.$$

<u>Möbius Trasformation</u>

Möbius transformations are special class of functions which naturally occur at various stages in complex analysis.

DEFINITION 1. A map $S(z) := \dfrac{az+b}{cz+d}$ is called a Möbius transformation if $ad - bc \neq 0$, where $a, b, c, d \in \mathbb{C}$.

EXAMPLE 1.

- $Id(z) = z$ is a Möbius transformation with $a = c = 1$ and $b = d = 0$.

- $S(z) = \dfrac{z-i}{z+i}$ is a Möbius transformation with $a = c = 1, b = -i$ and $d = i$. Here we can see that $S(z)$ is holomorphic at $\mathbb{C} \setminus \{-i\}$.

The observation we made on the last example can be stated in general as Möbius transformations are rational functions.
A Möbius transformation $S(z) = \dfrac{az+b}{cz+d}$ is holomorphic at $\mathbb{C} \setminus \left\{\dfrac{-d}{c}\right\}$.

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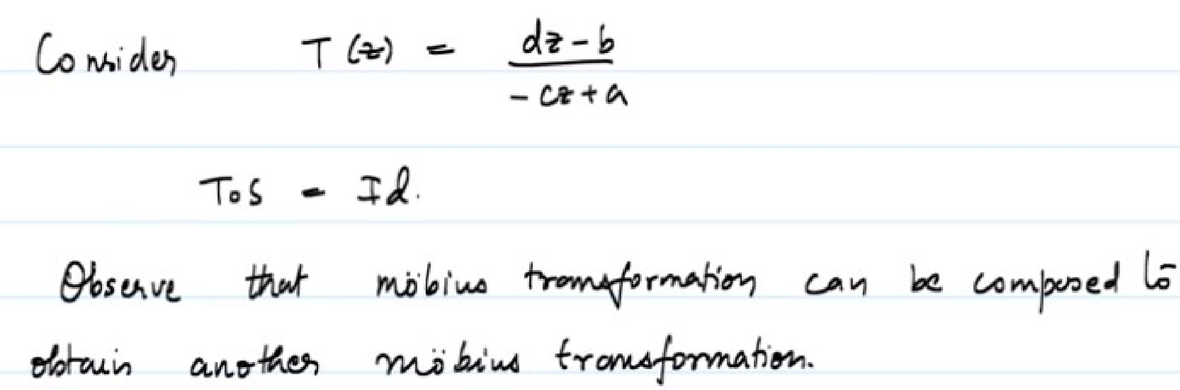

Consider $T(z) = \dfrac{dz-b}{-cz+a}$. Now at the domain of definition of T, it satisfies the condition that $T \circ S = Id$.

Observe that Möbius transformation can be composes to obtain another Möbius transformation. Hence Möbius transformations form a group.

We shall consider a Möbius transformation to be defined on $\hat{\mathbb{C}}$ rather than on $\mathbb{C}$ by defining for $S(z) = \dfrac{az+b}{cz+d}$,

$S\left(\dfrac{-d}{c}\right) = \infty$ and $S(\infty) = \dfrac{a}{c}$. When $c = 0$, then $S(\infty) = \infty$.

EXERCISE 2. S is a bijection on $\hat{\mathbb{C}}$.

EXAMPLE 3.

- Translation: $S(z) = z + b$, for $b \in \mathbb{C}$.
- Dilation: $S(z) = az$, for $a \neq 0$. For $a = e^{i\theta}$, S is an isometry on $\mathbb{C}$ where $\theta \in \mathbb{R}$.

- Inversion: $S(z) = \dfrac{1}{z}$.

PROPOSITION 4. *Any Möbius transformation can be written as a composition of translations, dilation and inverson.*

In the proposition, note that it is not necessary that all must appear. For example if you take a dilation, we doesn't need to compose with any translation and inversion.

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Proof: $S(z) = \dfrac{az+b}{cz+d}$.

If $c = 0$, $S(z) = (a/d)z + (b/d) = S_1 \circ S_2(z)$

where $S_2(z) = (a/d)z$ and $S_1(z) = z + (b/d)$.

PROOF. Consider any Möbius transformation $S(z) = \dfrac{az+b}{cz+d}$.

If $c = 0$, $S(z) = \left(\dfrac{a}{d}\right)z + \left(\dfrac{b}{d}\right) = S_1 \circ S_2(z)$, where $S_1(z) = z + \left(\dfrac{b}{d}\right)$ and $S_2(z) = \left(\dfrac{a}{d}\right)z$.

If $c \neq 0$, we can see that $S = S_4 \circ S_3 \circ S_2 \circ S_1$, where $S_1(z) = z + \left(\dfrac{d}{c}\right)$, $S_2(z) = \dfrac{1}{z}$, $S_3(z) = \dfrac{(bc-ad)}{c^2} z$ and $S_4(z) = z + \left(\dfrac{a}{c}\right)$. □

Let $S(z) = \dfrac{az+b}{cz+d}$ and w be a fixed point $S \implies S(w) = w \implies \dfrac{aw+b}{cw+d} = w \implies aw + b = w(cw+d) \implies cw^2 + (d-a)w - b = 0$. Hence if S is not the identity, then S can have a maximum of two fixed points.

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Let a, b, c be distinct pts. and α, β and γ be s.t $S(a) = \alpha$, $S(b) = \beta$ and $S(c) = \gamma$.

Let T be another möbius transf. s.t.
$T(a) = \alpha$, $T(b) = \beta$ and $T(c) = \gamma$.

Let a, b, c be distinct points in $\hat{\mathbb{C}}$ and α, β, and γ be such that $S(a) = \alpha, S(b) = \beta$ and $S(c) = \gamma$. Let T be any other Möbius transformation such that $T(a) = \alpha, T(b) = \beta$ and $T(c) = \gamma$. Then $T^{-1} \circ S$ is a Möbius transformation which fixes $a, b, c \in \hat{\mathbb{C}}$. But we have already observed that all non-identity Möbius transformation fixes maximum of two points, which forces $T^{-1} \circ S = Id \implies T = S$ since a, b, c are distinct points.

Let $a, b, c \in \hat{\mathbb{C}}$ be distinct.

If $a, b, c \in \mathbb{C}$, define

$$S(z) := \frac{\left(\frac{(z-a)}{(b-a)}\right)}{\left(\frac{(z-c)}{(b-c)}\right)}.$$

Then $S(a) = \infty, S(b) = 1$ and $S(c) = \infty$.

If $a = \infty$, then define

$$S(z) := \frac{(b-c)}{(z-c)}.$$

Then $S(\infty) = 0, S(b) = 1, S(c) = \infty$.

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If $b = \infty$,

$$S(z) = \frac{z-a}{z-c}$$

If $c = \infty$

$$S(z) = \frac{(z-a)}{(b-a)}.$$

If $b = \infty$, then

$$S(z) = \frac{(z-a)}{(z-c)}.$$

If $c = \infty$, then

$$S(z) = \frac{(z-a)}{(b-a)}.$$

Thus from the observations we made, we have the following proposition:

PROPOSITION 5. *Given distinct points $a, b, c \in \hat{\mathbb{C}}$, then there exists a unique Möbius transformation S such that $S(a) = 0, S(b) = 1$ and $S(c) = \infty$.*

COROLLARY 6. *Given distinct points $a, b, c \in \hat{\mathbb{C}}$ and distinct points $\alpha, \beta, \gamma \in \hat{\mathbb{C}}$, then there exists a unique Möbius transformation S such that $S(a) = \alpha, S(b) = \beta$ and $S(c) = \gamma$.*

PROOF. Let S_1 be a Möbius transformation such that $S_1(a) = 0, S_1(b) = 1$ and $S_1(c) = \infty$. Similarly, let S_2 be a Möbius transformation such that $S_2(\alpha) = 0, S_2(\beta) = 1$ and $S_2(\gamma) = \infty$. Now define, $T := S_2^{-1} \circ S_1$. Then T maps a to α, b to β and c to $\gamma \implies T = Id$.

$\square$

One of the important and also interesting property of Möbius transformation is, it maps generalized circles to generalized circles.

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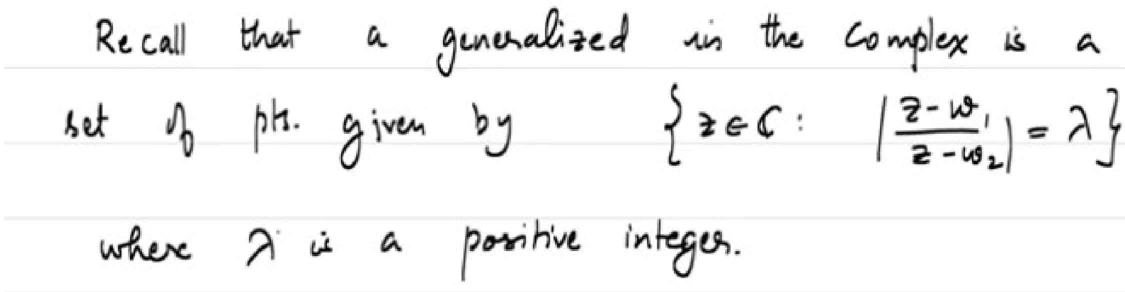

Recall that a generalized circle in the complex plane is a set of points given by $\left\{ z \in \mathbb{C} : \left| \dfrac{z-\omega_1}{z-\omega_2} \right| = \lambda \right\}$ where λ is a positive real number.

PROPOSITION 7. *Let S be a Möbius transformation. Then S maps the set $\mathbb{R} \cup \{\infty\}$ to a generalized circle.*

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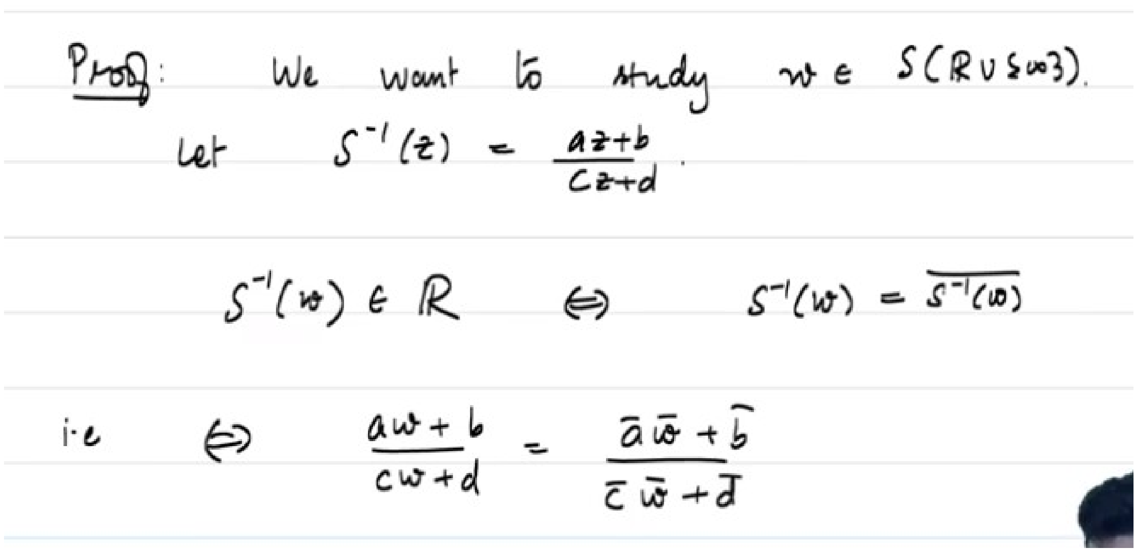

PROOF. We want to study $w \in S(\mathbb{R} \cup \{\infty\})$. Let $S^{-1}(z) = \dfrac{az+b}{cz+d}$. Now $S^{-1}(w) \in \mathbb{R}$

$$\iff S^{-1}(w) = \overline{S^{-1}(w)}$$

$$\iff \frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}$$

$$\Longleftrightarrow (a\bar{c}-\bar{a}c)|w|^2+(a\bar{d}-\bar{b}c)\,w+(b\bar{c}-\bar{a}d)\,\bar{w}+b\bar{d}-\bar{b}d=0$$

If $a\bar{c}=\bar{a}c$,

$$\alpha w-\bar{\alpha}\bar{w}=\bar{b}d-b\bar{d} \text{ where } \alpha=a\bar{d}-\bar{b}c$$

$$2i\mathfrak{Im}(\alpha w)=2i\mathfrak{Im}(\bar{b}d)$$

$$\mathfrak{Im}(\alpha w)=\mathfrak{Im}(\bar{b}d)=c\in\mathbb{R}.$$

If $a\bar{c}\neq\bar{a}c$

$$\Longleftrightarrow |w|^2+\alpha w+\bar{\alpha}\bar{w}+\gamma=0, \text{ where } \alpha=\frac{a\bar{d}-\bar{b}c}{a\bar{c}-\bar{a}c}, \gamma=\frac{b\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}c}$$

$$\Longleftrightarrow |w-\alpha|^2=|\alpha|^2-\gamma>0 \text{ (Why?)}.$$

□

THEOREM 8. *A Möbius transformation maps generalized circle to generalized circles.*

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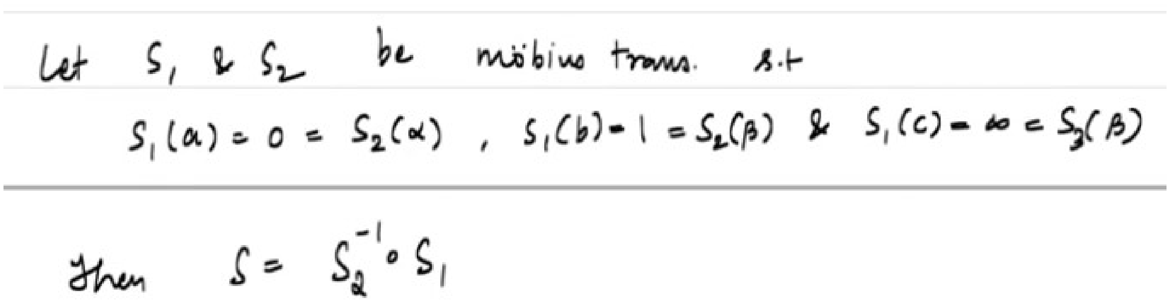

PROOF. Pick three distinct points a, b, c on the given generalized circle. Let S be a Möbius transformation and $\alpha = S(a), \beta = S(b)$ and $\gamma = S(c)$.

Let S_1 and S_2 be Möbius transformations such that $S_1(a) = 0 = S_2(\alpha), S_1(b) = 1 = S_2(\beta)$ and $S_1(c) = \infty = S_2(\gamma)$. Then $S = S_2^{-1} \circ S_1$. S_1 maps a generalized circle containing a, b, c to $\mathbb{R}\cup\{\infty\}$ and by the previous proposition S_2^{-1} maps $\mathbb{R}\cup\{\infty\}$ to a generalized circle. □