Complex Analysis

Prof. Pranav Haridas

Kerala School of Mathematics

Module No - 3

Lecture No – 13

Harmonic Functions

In the last week we explore the Cauchy–Riemann equations where the right set of conditions which ensured that a differentiable function also turned out to be complex differentiable. Given a holomorphic function, the Cauchy–Riemann equations tied the real part of the holomorphic function with imaginary part of the holomorphic function. We begin this week by observing that the Cauchy–Riemann equations imparts an extra set of condition on these functions, real part and the imaginary part of a given holomorphic function which is captured by the more common harmonic functions.

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C-R eqn tells us that $f = u + iv$, then

$$\left.\begin{aligned} &\partial u/\partial x\,(z) = \partial v/\partial y\,(z) \\ \text{and} \quad &\partial u/\partial y\,(z) = -\,\partial v/\partial x\,(z) \end{aligned}\right\} \longrightarrow (*)$$

From $*$, we get

$$\partial^2 u/\partial x^2\,(z) = \partial^2 v/\partial x \partial y\,(z)$$

$$\partial^2 u/\partial y^2\,(z) = -\frac{\partial^2 v}{\partial y \partial x}(z).$$

In this lecture we will always assume that our holomorphic functions are twice continuously differentiable. So I would like to note that this is a redundant condition. It is redundant because holomorphic functions will always satisfy this criteria. We will prove that later. But since we have not proved it yet let me just put this extra condition.

Let $f : U \to \mathbb{C}$ be a holomorphic function. The Cauchy- Riemann equation tells us that if $f = u + iv$, then

$$\left.\begin{aligned} \frac{\partial u}{\partial x}(z) &= \frac{\partial v}{\partial y}(z) \\ \frac{\partial u}{\partial y}(z) &= -\frac{\partial v}{\partial x}(z) \end{aligned}\right\} \to (*)$$

From $(*)$, we get

$$\frac{\partial^2 u}{\partial x^2}(z) = \frac{\partial^2 v}{\partial x \partial y}(z)$$

$$\frac{\partial^2 u}{\partial y^2}(z) = -\frac{\partial^2 v}{\partial y \partial x}(z).$$

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$$\partial^2 u/\partial x^2 (z) = \partial v/\partial x \partial y (z)$$

$$\partial^2 u/\partial y^2 (z) = -\frac{\partial^2 v}{\partial y \partial x}(z).$$

By Clairaut's theorm,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

and hence we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

By Clairaut's theorem,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

and hence we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Let $\Delta := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$. Then Δ is called the Laplacian operator.

$$\Delta u = \Delta v = 0 \implies \Delta f = 0.$$

DEFINITION 1. Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function. We say u is harmonic if $\Delta u = 0$.

EXAMPLE 1.

- Given any holomorphic function f. Consider $\mathfrak{Re}(f)$ and $\mathfrak{Im}(f)$, both will be harmonic functions.
- Consider $f(z) = \bar{z}$. Then f is a harmonic function which is not holomorphic.
- Let $u(z) = x^2 - y^2$, which then will be harmonic.

Recall that, given $\Omega \subset X$, where X is a metric space, the boundary of Ω, $\partial\Omega$ is given by $\partial\Omega = \overline{\Omega} \cap \overline{X \setminus \Omega}$.

EXAMPLE 2. $\partial B(x_0, r) = \{x \in X : d(x, x_0) = r\}$.

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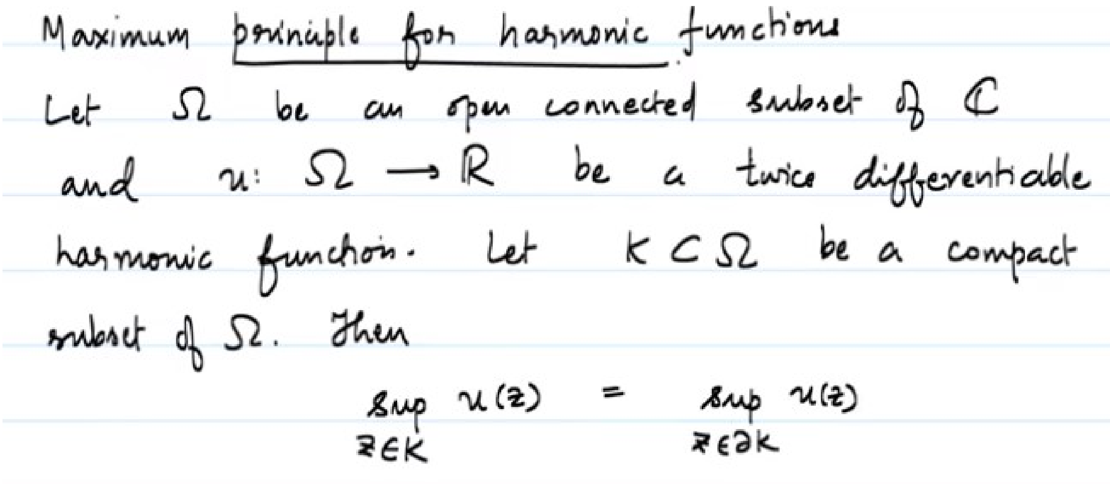

THEOREM 3 (**Maximum Principle for Harmonic Functions**). *Let Ω be an open connected subset of $\mathbb{C}$ and $u : \Omega \to \mathbb{R}$ be a twice differentiable harmonic function. Let $K \subset \Omega$ be a compact subset of Ω. Then,*

$$\sup_{z\in K} u(z) = \sup_{z\in\partial K} u(z)$$

and

$$\inf_{z\in K} u(z) = \inf_{z\in\partial K} u(z)$$

PROOF. We shall prove that

$$\sup_{z\in K} u(z) = \sup_{z\in\partial K} u(z).$$

(By considering $-u$ which is also a harmonic function, the second statement follows.)

We already have

$$\sup_{z\in\partial K} u(z) \le \sup_{z\in K} u(z).$$

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Assume

$$\sup_{z \in \partial K} u(z) < \sup_{z \in K} u(z).$$

Let z_0 be a pt. in K s.t

$$u(z_0) = \sup_{z \in K} u(z)$$

Assume that

$$\sup_{z \in \partial K} u(z) < \sup_{z \in K} u(z).$$

Let z_0 be a point in K such that $u(z_0) = \sup_{z \in K} u(z)$.

Since z_0 attains the maximum, we have $\frac{\partial^2 u}{\partial x^2}(z_0) \leq 0$ and $\frac{\partial^2 u}{\partial y^2}(z_0) \leq 0$. Let $\delta = \sup_{z \in K} u(z) - \sup_{z \in \partial K} u(z)$.

Consider the function $x^2 + y^2$, then this is bounded above in K by a constant M, as K is compact.

Define $u_\epsilon = u(z) + \epsilon(x^2 + y^2)$, where $\epsilon < \frac{\delta}{2M}$.

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Check that $\sup_{z \in \partial K} u_\varepsilon(z) < \sup_{z \in K} u_\varepsilon(z)$.

Let z_ε be a pt. in K where u_ε attains its maximum.

Now it is left as an exercise to reader to check that $\sup\limits_{z\in\partial K} u_\epsilon(z) < \sup\limits_{z\in K} u_\epsilon(z)$.

Let z_ϵ be a point in K where u_ϵ attains its maximum.

Hence

$$\left.\begin{array}{l} \dfrac{\partial^2 u_\epsilon}{\partial x^2}(z_\epsilon) \leq 0 \\ \dfrac{\partial^2 u_\epsilon}{\partial y^2}(z_\epsilon) \leq 0 \end{array}\right\} \rightarrow (**)$$

But we know that,

$$\frac{\partial^2 u_\epsilon}{\partial x^2} + \frac{\partial^2 u_\epsilon}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + 2\epsilon + \frac{\partial^2 u}{\partial y^2} + 2\epsilon$$

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$$\frac{\partial^2 u_\varepsilon}{\partial x^2} + \frac{\partial^2 u_\varepsilon}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + 2\varepsilon + \frac{\partial^2 u}{\partial y^2} + 2\varepsilon$$

$$\Delta u_\varepsilon(z_\varepsilon) = \cancel{\Delta u(z_\varepsilon)} + 4\varepsilon$$

$$\Rightarrow \quad \Delta u_\varepsilon(z_\varepsilon) > 0$$

which is a contradiction to $(**)$.

And the Laplacian of u_ϵ,

$$\Delta u_\epsilon(z_\epsilon) = \Delta u(z_\epsilon) + 4\epsilon \implies \Delta u_\epsilon(z_\epsilon) > 0$$

which is a contradiction to $(**)$.

Hence

$$\sup_{z\in K} u(z) = \sup_{z\in\partial K} u(z).$$

$\square$

THEOREM 4 (**Maximum Principle for Holomorphic Functions**). *Let $U \subseteq \mathbb{C}$ be open and connected and let $f : U \to \mathbb{C}$ be a holomorphic function. Then for $K \subset U$ compact, we have*

$$\sup_{z\in K} |f(z)| = \sup_{z\in\partial K} |f(z)|.$$

PROOF. Let $z \in \mathbb{C}$, then

$$|z| = \sup_{\theta\in\mathbb{R}} \mathfrak{Re}(ze^{i\theta})$$

$$\begin{aligned}\sup_{z\in K} |f(z)| &= \sup_{z\in K} \sup_{\theta\in\mathbb{R}} \mathfrak{Re}\left(f(z)e^{i\theta}\right)\\ &= \sup_{\theta\in\mathbb{R}} \sup_{z\in K} \mathfrak{Re}\left(f(z)e^{i\theta}\right).\end{aligned}$$

Define $f_\theta(z) = e^{i\theta} f(z)$. Then f_θ is holomorphic. Hence $\mathfrak{Re}(f_\theta)$ is harmonic.

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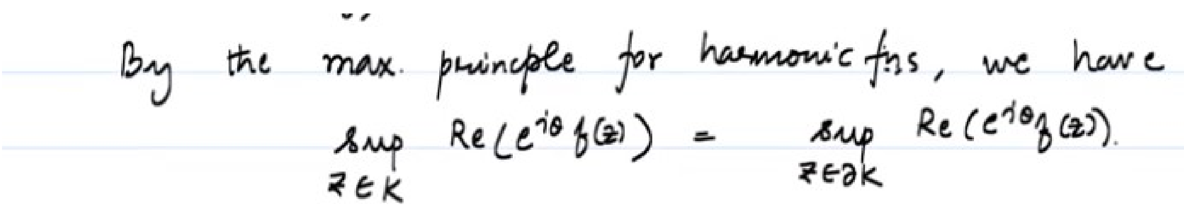

By the maximum principle for harmonic functions, we have

$$\sup_{z\in K} \mathfrak{Re}\left(e^{i\theta} f(z)\right) = \sup_{z\in\partial K} \mathfrak{Re}\left(e^{i\theta} f(z)\right).$$

Hence, we then have

$$\begin{aligned}\sup_{z\in K} |f(z)| &= \sup_{\theta\in\mathbb{R}} \sup_{z\in\partial K} \mathfrak{Re}\left(e^{i\theta} f(z)\right)\\ &= \sup_{z\in\partial K} \sup_{\theta\in\mathbb{R}} \mathfrak{Re}\left(e^{i\theta} f(z)\right)\\ &= \sup_{z\in\partial K} |f(z)|.\end{aligned}$$

$\square$

DEFINITION 2 (**Harmonic Conjugate**). Let $u : \Omega \subseteq \mathbb{C} \to \mathbb{R}$ be a twice differentiable harmonic function. We say that $v : \Omega \to \mathbb{R}$ is a harmonic conjugate of u if $f := u + iv$ is holomorphic.

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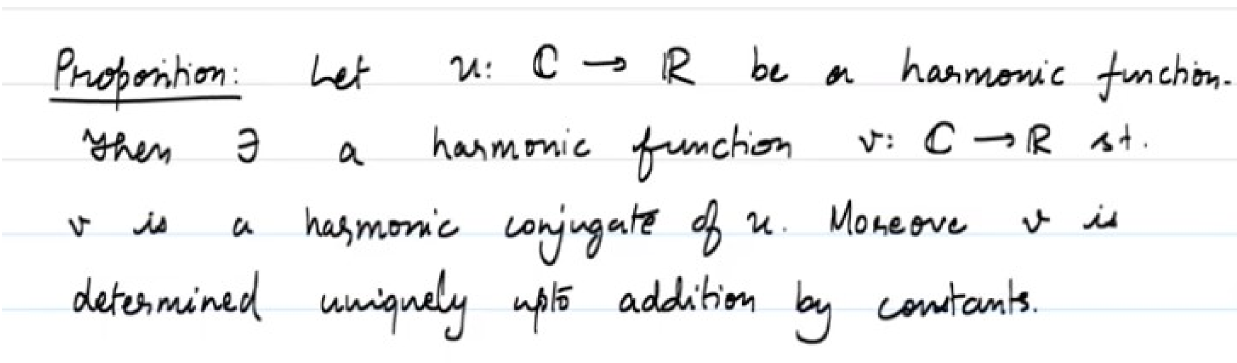

PROPOSITION 5. *Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function. Then there exists a harmonic function $v : \mathbb{C} \to \mathbb{R}$ such that v is a harmonic conjugate of u. Moreover v is determined uniquely up to addition by constants.*

PROOF. First we may assume the existence of such a harmonic conjugate and prove the uniqueness. That is, we will prove if there exists two harmonic conjugates, v_1 and v_2, then $v_1 - v_2$ is a constant.

Let v_1 be a harmonic conjugate. By using fundamental theorem of calculus, we have

$$\begin{aligned} v_1(x+iy) &= v_1(0) + \int_0^x \frac{\partial v_1}{\partial x}(t)\,dt + \int_0^y \frac{\partial v_1}{\partial y}(x+it)\,dt \\ &= v_1(0) - \int_0^x \frac{\partial u}{\partial y}(t)\,dt + \int_0^y \frac{\partial u}{\partial x}(x+it)\,dt. \end{aligned}$$

Similarly, if v_2 is any other harmonic conjugate of u, then

$$v_2(x+iy) = v_2(0) - \int_0^x \frac{\partial u}{\partial y}(t)\,dt + \int_0^y \frac{\partial u}{\partial x}(x+it)\,dt.$$

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Hence $\quad v_1(x+iy) - v_2(x+iy) = v_1(0) - v_2(0) = c \in \mathbb{C}.$

Existence:

Define $\quad v(x+iy) = -\int_0^x \frac{\partial u}{\partial y}(t)\,dt + \int_0^y \frac{\partial u}{\partial x}(x+it).$

Hence,

$$v_1(x+iy) - v_2(x+iy) = v_1(0) - v_2(0) = c \in \mathbb{C}.$$

Now let us prove the existence part of the proposition.

Define

$$v(x+iy) = -\int_0^x \frac{\partial u}{\partial y}(t)\,dt + \int_0^y \frac{\partial u}{\partial x}(x+it)\,dt.$$

Then

$$\frac{\partial v}{\partial y}(x+it) = \frac{\partial u}{\partial x}(x+it)$$

which is one of the expressions in the Cauchy-Riemann equations.

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$$= -\frac{\partial u}{\partial y}(x) + \int_0^y \frac{\partial^2 u}{\partial x^2}(x+it)\,dt.$$

Since u is harmonic, we have

$$\int_0^y \frac{\partial^2 u}{\partial x^2}(x+it)\,dt = -\int_0^y \frac{\partial^2 u}{\partial y^2}(x+it)\,dt.$$

$$\begin{aligned}\frac{\partial v}{\partial x}(x+iy) &= -\frac{\partial u}{\partial y}(x) + \frac{\partial}{\partial x}\left(\int_0^y \frac{\partial u}{\partial x}(x+it)\,dt\right)\\ &= -\frac{\partial u}{\partial y}(x) + \int_0^y \frac{\partial^2 u}{\partial x^2}(x+it)\,dt.\end{aligned}$$

Since u is harmonic, we have

$$\begin{aligned}\int_0^y \frac{\partial^2 u}{\partial x^2}(x+it)\,dt &= -\int_0^y \frac{\partial^2 u}{\partial y^2}(x+it)\,dt\\ &= -\left(\frac{\partial u}{\partial y}(x+iy) - \frac{\partial u}{\partial y}(x)\right).\end{aligned}$$

$$\frac{\partial v}{\partial x}(x+iy) = -\frac{\partial u}{\partial y}(x) - \frac{\partial u}{\partial y}(x+iy) + \frac{\partial u}{\partial y}(x) = -\frac{\partial u}{\partial y}(x+iy).$$

Hence u, v satisfy the Cauchy- Riemann equations and therefore $f = u + iv$ is holomorphic. $\square$