**Complex Analysis**

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**Lecture No – 3.3**

**Problem Session**

PROBLEM 1. Let $T : \mathbb{C} \to \mathbb{C}$ be an isometry of the complex plane. Then show that $T$ is given by either

$$T(z) = z_0 + wz \qquad \text{for} \qquad z \in \mathbb{C}$$

or

$$T(z) = z_0 + w\bar{z} \qquad \text{for} \qquad z \in \mathbb{C}$$

where $z_0 \in \mathbb{C}$ is fixed and $w \in S^1$.

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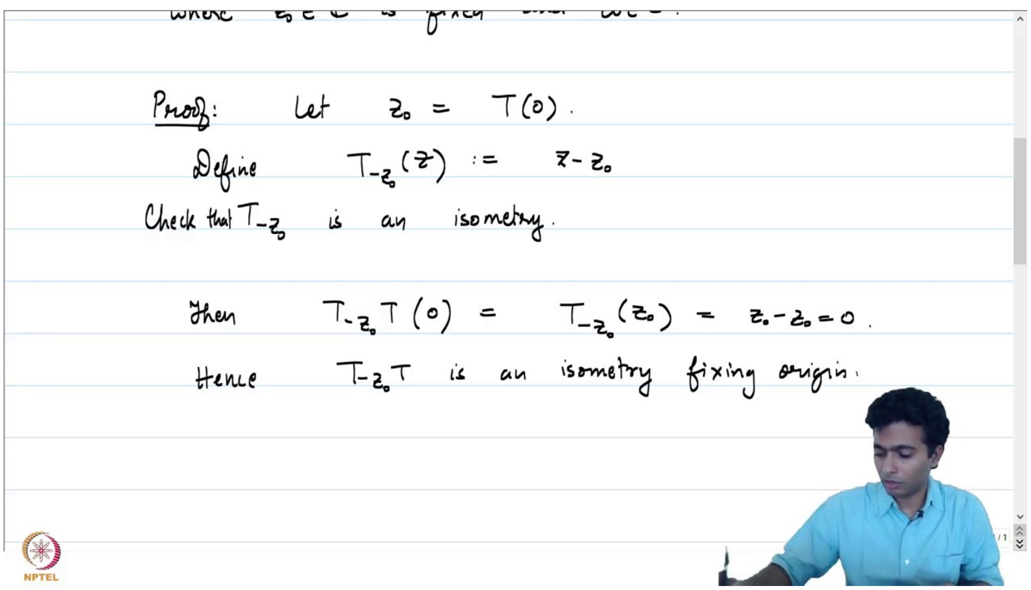


SOLUTION 1. Let $z_0 = T(0)$. Define $T_{-z_0} := z - z_0$. Now it is left to reader to verify that $T_{-z_0}$ is an isometry. Then $T_{-z_0}T(0) = T_{-z_0}(z_0) = z_0 - z_0 = 0$. Hence $T_{-z_0}T$ is an isometry fixing origin.

Then by the theorem proved earlier, we have either

$$T_{-z_0}T(z) = wz \quad \text{for} \quad z \in \mathbb{C}, \text{ (where } w \in S^1)$$

or

$$T_{-z_0}T(z) = w\bar{z} \quad \text{for} \quad z \in \mathbb{C}, \text{ (where } w \in S^1).$$

Define $T_{z_0}(z) = z + z_0$. Then observe that $T_{z_0}$ is also an isometry. Moreover $T_{z_0}T_{-z_0}(z) = T_{z_0}(z - z_0) = z = id(z) \implies T_{z_0}T_{-z_0} = id$.

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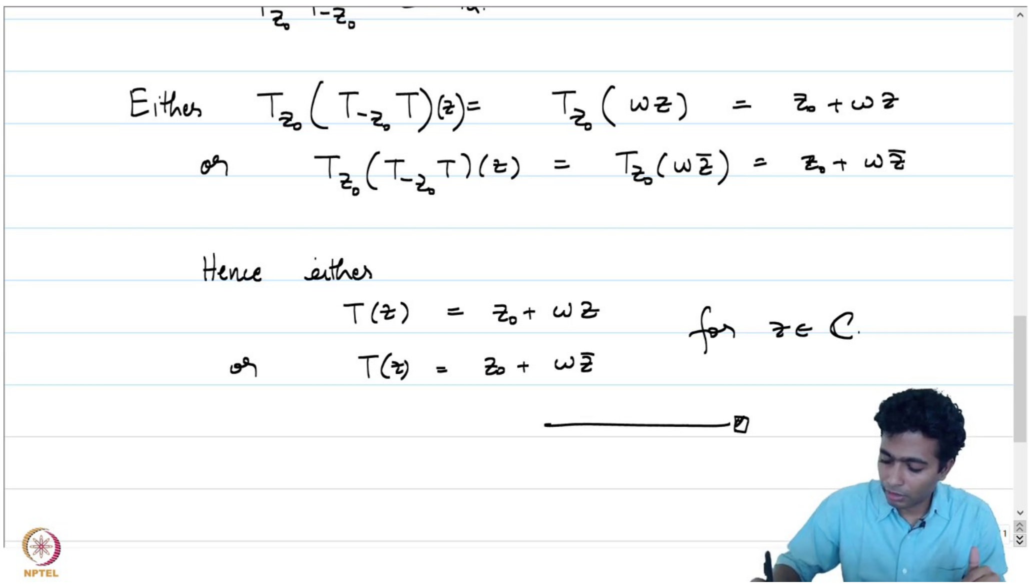


Hence, either

$$T_{z_0}(T_{-z_0}T)(z) = T_{z_0}(wz) = z_0 + wz$$

or

$$T_{z_0}(T_{-z_0}T)(z) = T_{z_0}(w\bar{z}) = z_0 + w\bar{z}.$$

We know that function composition is associative, hence $T_{z_0}(T_{-z_0}T) = (T_{z_0}T_{-z_0})T = idT = T$.

Hence for $z \in \mathbb{C}$ either

$$T(z) = z_0 + wz$$

or

$$T(z) = z_0 + w\bar{z}.$$

PROBLEM 2. Let $n$ be a positive integer. Show that the complex number solutions to the equation $z^n = 1$ are given by $z_k = e^{\frac{2\pi i k}{n}}$ for $k = 0, 1, \ldots, n-1$. These roots are called the $n^{\text{th}}$ roots of unity.

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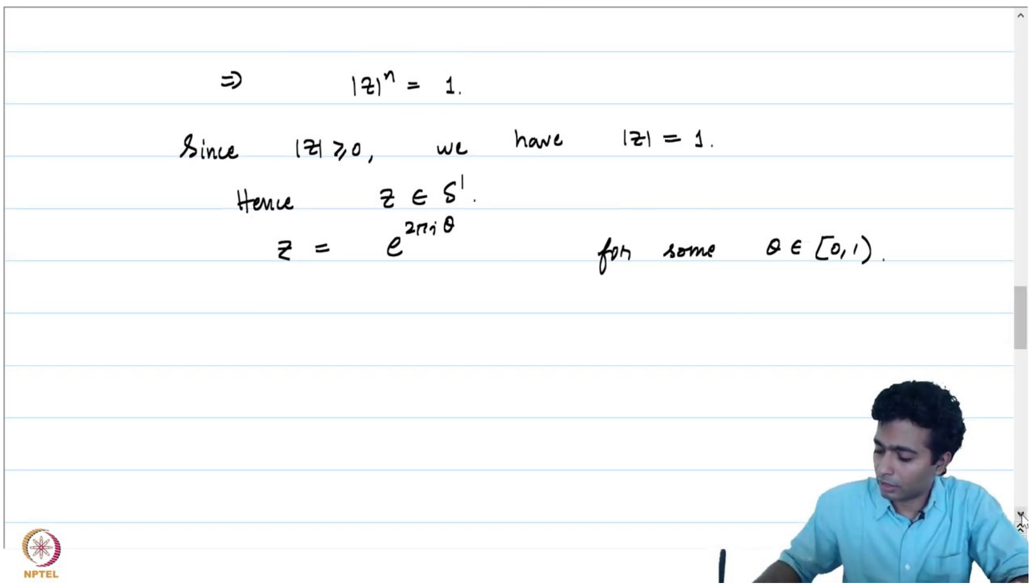


SOLUTION 2. Consider $z^n = 1$. Then $|z^n| = 1 \implies |z|^n = 1$. Since $|z| \geq 0$, we have $|z| = 1$. Hence $z \in S^1 \implies z = e^{2\pi i\theta}$ for some $\theta \in [0,1)$.

$$z^n = 1 \implies e^{2\pi i(n\theta)} = 1$$

$$\implies 2\pi(n\theta) = 2\pi k, \text{ where } k \in \mathbb{Z}$$

$$\implies \theta = \frac{k}{n}, \text{ where } k \in \mathbb{Z}.$$

Notice that if $\ell_1$ and $\ell_2$ are such that

$$e^{2\pi i\frac{\ell_1}{n}} = e^{2\pi i\frac{\ell_2}{n}} \implies e^{2\pi i\frac{(\ell_1-\ell_2)}{n}} = 1 \implies \frac{\ell_1 - \ell_2}{n} \in \mathbb{Z}.$$

Thus if $\ell_1 = \ell_2 + kn$, then $e^{2\pi i \frac{\ell_1}{n}} = e^{2\pi i \frac{\ell_2}{n}}$.
Hence every complex number $z$ such that $z^n = 1$ is given by $e^{\frac{2\pi i k}{n}}$ where $k = 0, 1, \dots n-1$.
(Any other integer $\ell$ can be written as $\ell = dn + k$.)

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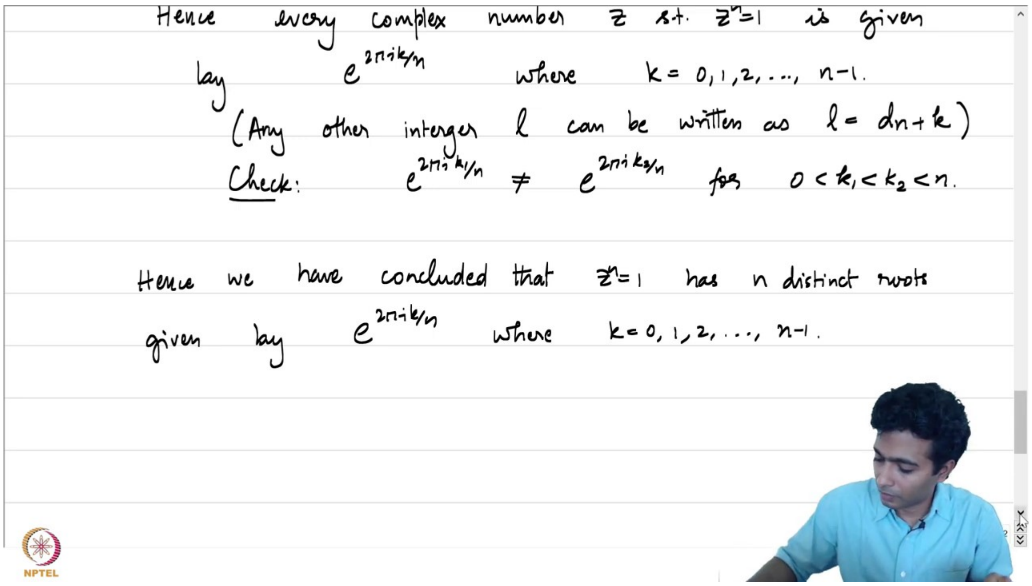


Now it is left for reader to check that $e^{\frac{2\pi i k_1}{n}} \neq e^{\frac{2\pi i k_2}{n}}$ for $0 < k_1 < k_2 < n$.
Hence we have concluded that $z^n = 1$ has $n$ distinct roots given by $e^{\frac{2\pi i k}{n}}$ where $k = 0, 1, \dots, n-1$ and are called the $n^{\text{th}}$ roots of unity.

PROBLEM 3. Show that if $w$ is a non-zero complex number and $n$ a positive integer, then there exists $n$ distinct roots of the equation $z^n = w$. Any two roots differ by multiplication by a root of unity.

SOLUTION 3. Consider $z^n = w$. In polar coordinates, let $w = re^{i2\pi\theta}$. Then $|z|^n = |w| = r \implies |z| = r^{1/n}$. If $z = r^{1/n} e^{i2\pi\varphi}$

$$z^n = w$$

$$\implies \left(r^{1/n} e^{i2\pi\varphi}\right)^n = re^{i2\pi\theta}$$

$$\implies e^{i2\pi(n\varphi - \theta)} = 1$$

$$\implies n\varphi - \theta = k \in \mathbb{Z}$$

$$\implies \varphi = \frac{k+\theta}{n} \text{ where } k \in \mathbb{Z}.$$

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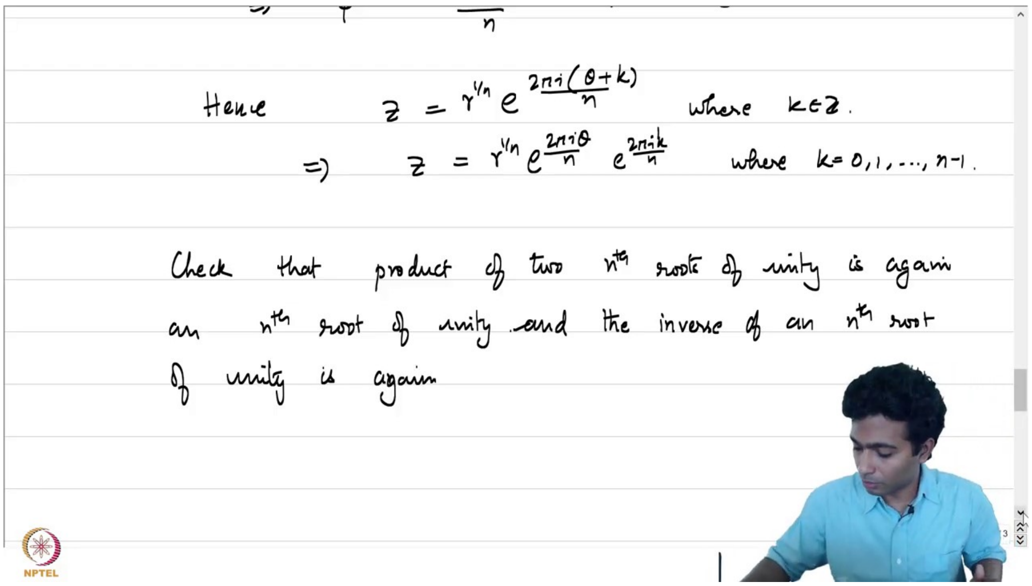


Hence $z = r^{1/n} e^{\frac{i2\pi(\theta+k)}{n}}$ where $k \in \mathbb{Z} \implies z = r^{1/n} e^{\frac{i2\pi\theta}{n}} \cdot e^{\frac{i2\pi k}{n}}$ where $k = 0, 1, \ldots n-1$. Check that product of two $n^{\text{th}}$ roots of unity is again an $n^{\text{th}}$ root of unity and the inverse of an $n^{\text{th}}$ root of unity is again an $n^{\text{th}}$ root of unity.

PROBLEM 4. Let $w_1, w_2$ be distinct complex numbers, and let $\lambda$ be a positive real number.

(i) Show that the set $\left\{ z : \frac{|z-w_1|}{|z-w_2|} = \lambda \right\}$ gives a circle when $\lambda \neq 1$ and a straight line when $\lambda = 1$.

(ii) Conversely, show that every circle and straight line can be written in this manner.

SOLUTION 4. Consider $\frac{|z-w_1|}{|z-w_2|} = \lambda$. Then,

$$|z-w_1|^2 = \lambda^2 |z-w_2|^2$$

$$|z|^2 - z\bar{w}_1 - \bar{z}w_1 + |w_1|^2 = \lambda^2(|z|^2 - z\bar{w}_2 - \bar{z}w_2 + |w_2|^2)$$

If $\lambda = 1$,

$$z(\bar{w}_2 - \bar{w}_1) + \bar{z}(\bar{w}_2 - \bar{w}_1) = |w_2|^2 - |w_1|^2$$

$$\bar{\alpha}z + \alpha\bar{z} = c, \text{ where } c \in \mathbb{R}$$

which is a line.

When $\lambda \neq 1$,

$$(1-\lambda^2)|z|^2 - z(\bar{w}_1 - \lambda^2 \bar{w}_2) - \bar{z}(w_1 - \lambda^2 w_2) = \lambda^2|w_2|^2 - |w_1|^2$$

$$|z|^2 - \bar{\alpha}z - \alpha\bar{z} + |\alpha|^2 = \frac{\lambda^2|w_2|^2 - |w_1|^2}{(1-\lambda^2)} + |\alpha|^2$$

$$|z-\alpha|^2 = r^2, \text{ where } r \in \mathbb{R}$$

which is a circle.

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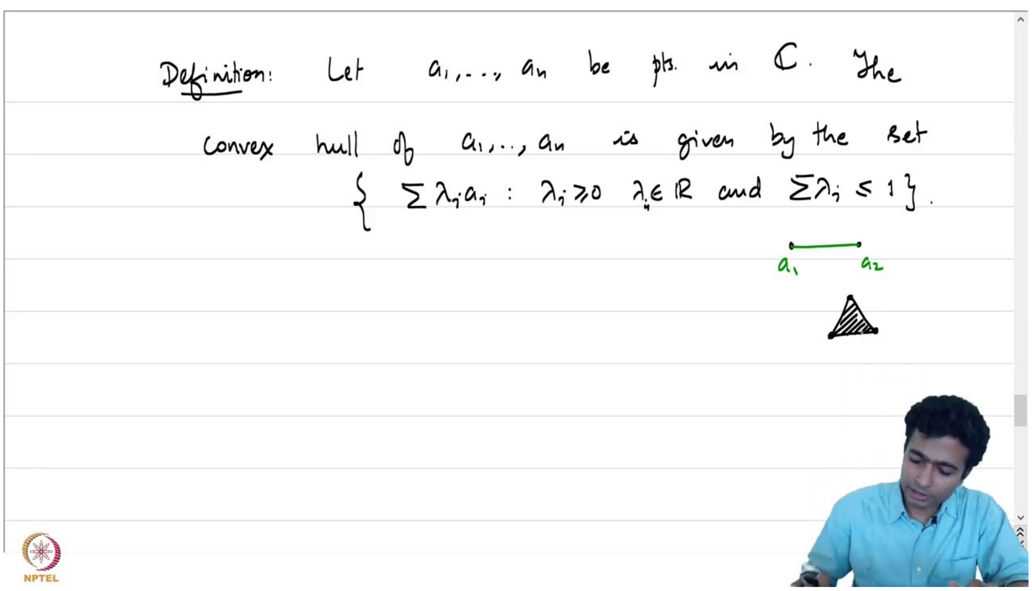


Converse part is left as an exercise to reader.

**<u>Definition:</u>** Let $a_1, a_2, \ldots, a_n \in \mathbb{C}$. The convex hull of $a_1, a_2, \ldots, a_n$ is given by the set $\{\sum \lambda_i a_i : \lambda_i \in \mathbb{R}, \lambda_i \geq 0 \text{ and } \sum \lambda_i = 1\}$.

PROBLEM 5. Let $P(z)$ be a polynomial given by $P(z) = c(z-z_1)(z-z_2)\cdots(z-z_n)$ where $z_1, z_2, \ldots z_n \in \mathbb{C}$(not necessarily distinct). Then the roots of $P'(z)$ lies in the convex hull of $z_1, z_2, \ldots z_n$.

SOLUTION 5. By product rule, $P'(z) = \sum_{j=1}^{n} c(z-z_1)\cdots\widehat{(z-z_j)}\cdots(z-z_n)$, here the hat indicates absence of the particular term.

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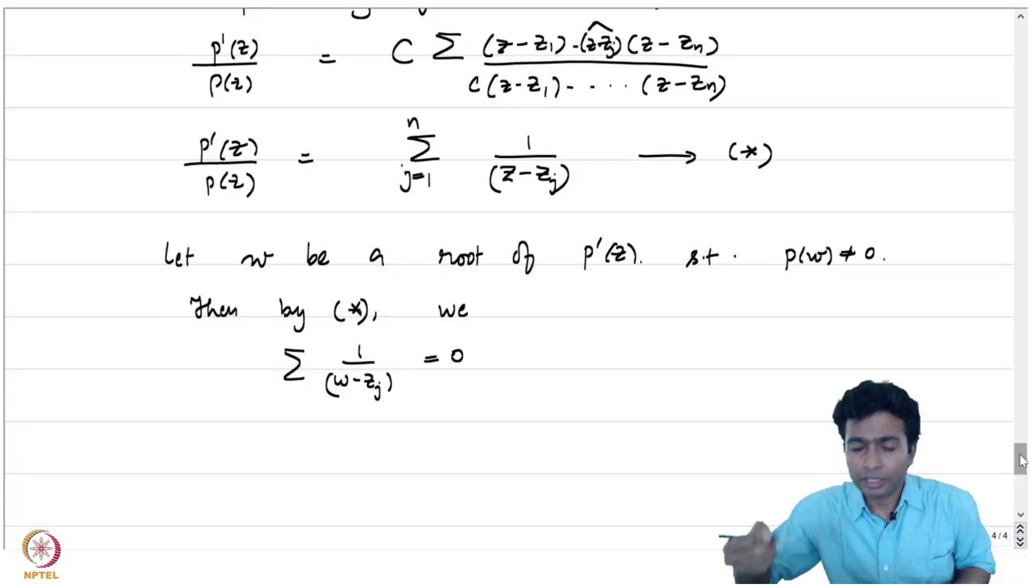


At the points away from the roots of $P$, we have,

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{c(z-z_1)\cdots\widehat{(z-z_j)}\cdots(z-z_n)}{c(z-z_1)(z-z_2)\cdots(z-z_n)} = \sum_{j=1}^{n} \frac{1}{(z-z_j)}. \rightarrow (*)$$

Let $w$ be a root of $P'(z)$ such that $P(w) \neq 0$. Then by $(*)$, we have

$$\sum_{j=1}^{n} \frac{1}{w-z_j} = 0$$

$$\implies \sum_{j=1}^{n} \frac{\bar{w}-\bar{z}_j}{|w-z_j|^2} = 0$$

$$\implies \bar{w}\sum \frac{1}{|w-z_j|^2} = \sum \frac{\bar{z}_j}{|w-z_j|^2}$$

$$w = \sum \frac{\frac{1}{|w-z_j|^2}}{\left(\sum \frac{1}{|w-z_j|^2}\right)} z_j.$$

Put $\frac{\frac{1}{|w-z_j|^2}}{\left(\sum \frac{1}{|w-z_j|^2}\right)} = \lambda_j$, then $\sum \lambda_j = 1$ and hence $w = \sum \lambda_j z_j$.