Complex Analysis

Prof. Pranav Haridas

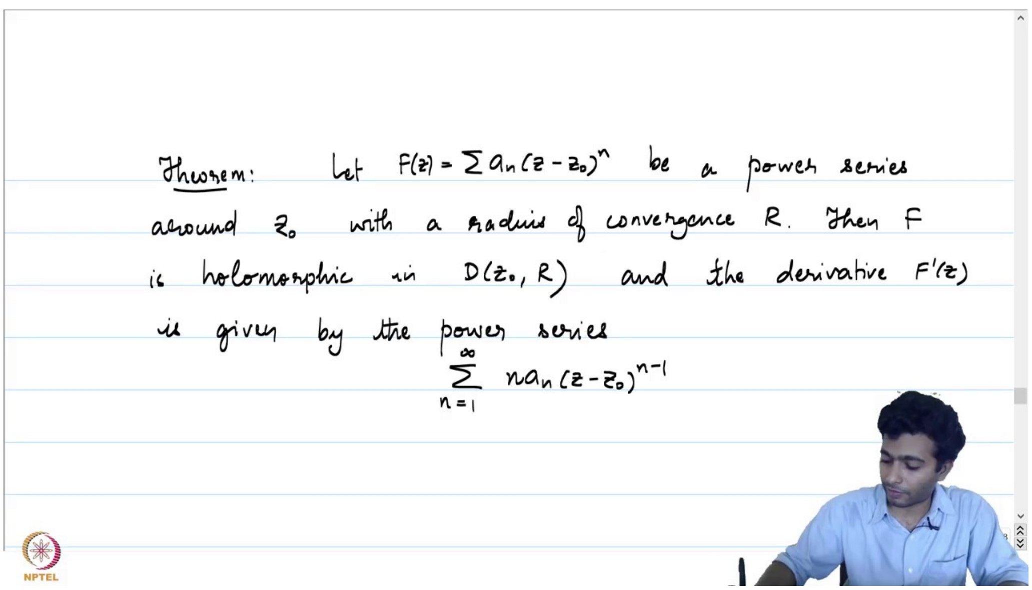
Kerala School of Mathematics

Lecture No – 3.2

Differentiation of Power Series

In the last lecture we defined what is meant by a power series and we proved that a power series converges absolutely in its disk of convergence we also remarked that a power series behaves very similar to polynomials in its disc of convergence. We begin this lecture by proving that a power series is holomorphic in its disc of convergence and that just like in the case of polynomials we can obtain the derivative by differentiating the power series term by term.

(Refer Slide Time: 01:11)



THEOREM 1. *Let $F(z) = \sum a_n(z - z_0)^n$ be a power series around z_0 with a radius of convergence R. Then F is holomorphic in $D(z_0, R)$ and the derivative $F'(z)$ is given by the power series,*

$$\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

which has a radius of convergence R.

PROOF. Consider the power series

$$\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z-z_0)^n.$$

Then $R' = \liminf_{n\to\infty} ((n+1)|a_{n+1}|)^{-1/n}$.
Now it is left to reader as an exercise to verify that $\lim_{n\to\infty} (n+1)^{1/n} = 0$.

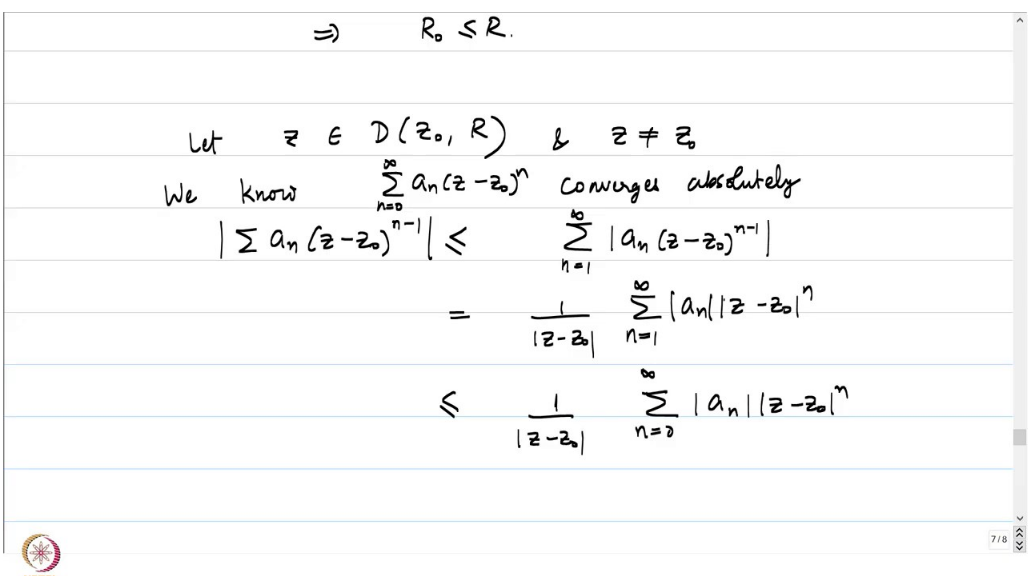
Claim: Let $R_0 = \liminf_{n\to\infty} (|a_{n+1}|)^{-1/n}$. Then $R_0 = R$.

For $z \in D(z_0, R_0)$,

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + (z-z_0) \sum_{n=1}^{\infty} a_n (z-z_0)^{n-1}$$

Hence, $\sum_{n=0}^{\infty} |a_n(z-z_0)^n| \le |a_0| + |(z-z_0)| \sum_{n=1}^{\infty} |a_n||(z-z_0)|^{n-1} \implies \sum a_n (z-z_0)^n$ converges in $D(z_0, R_0) \implies R_0 \le R$.

(Refer Slide Time: 07:56)



Let $z \in D(z_0, R)$ and $z \neq z_0$. We know $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely.

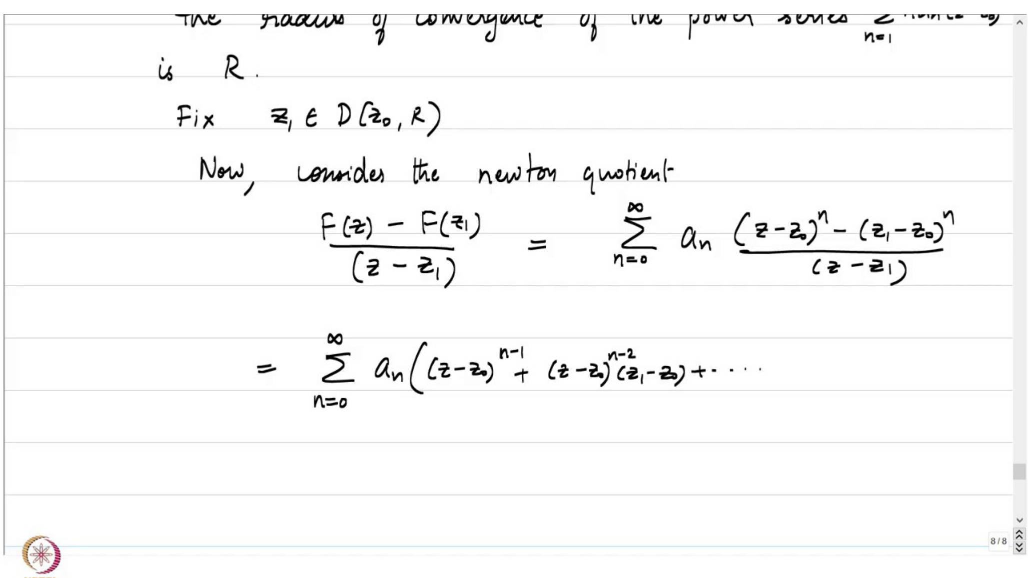
$$\begin{aligned}
\left| \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} \right| &\leq \sum_{n=1}^{\infty} |a_n (z - z_0)^{n-}| \\
&= \frac{1}{|z - z_0|} \sum_{n=1}^{\infty} |a_n||(z - z_0)|^n \\
&\leq \frac{1}{|z - z_0|} \sum_{n=1}^{\infty} |a_n||(z - z_0)|^n
\end{aligned}$$

Hence $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $D(z_0, R) \implies R \leq R_0$. Hence $R = R_0$.

Exercise: Let a_n and b_n be positive real numbers such that $a_n \to a$ and where $a > 0$ and $\liminf_{n\to\infty} b_n = b$. Then $\liminf_{n\to\infty} a_n b_n = ab$.

By using this exercise, we can conclude that the radius of convergence of the power series $\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ is R.

(Refer Slide Time: 15:25)



Fix $z_1 \in D(z_0, R)$. Now, consider the Newton quotient,

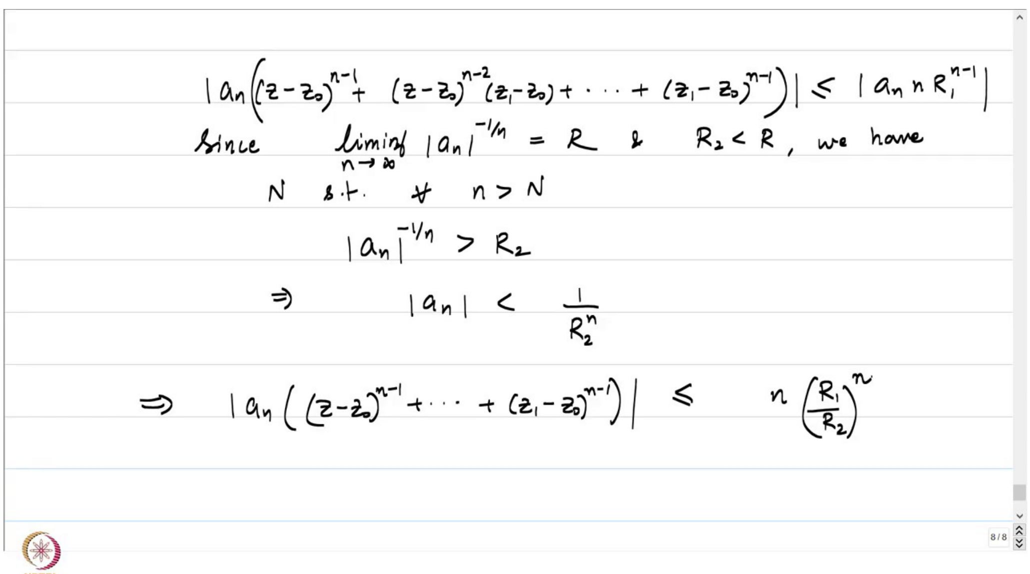
$$\frac{F(z)-F(z_1)}{(z-z_1)} = \sum_{n=0}^{\infty} a_n \frac{(z-z_0)^n-(z_1-z_0)^n}{(z-z_1)}$$
$$= \sum_{n=0}^{\infty} a_n \left((z-z_0)^{n-1}+(z-z_0)^{n-2}(z_1-z_0)+\cdots+(z_1-z_0)^{n-1}\right).$$

We want to show that,

$$\lim_{\substack{z\to z_1 \\ z\in D(z_0,R)\setminus\{z_1\}}} \sum_{n=1}^{\infty} a_n \left((z-z_0)^{n-1}+(z-z_0)^{n-2}(z_1-z_0)+\cdots+(z_1-z_0)^{n-1}\right) = \sum_{n=1}^{\infty} n a_n (z_1-z_0)^{n-1}.$$

Define $G(z) = \sum_{n=1}^{\infty} a_n \left((z-z_0)^{n-1}+(z-z_0)^{n-2}(z_1-z_0)+\cdots+(z_1-z_0)^{n-1}\right)$.
Let R_1, R_2 be such that $|z_1-z_0| < R_1 < R_2 < R$ and $\varepsilon > 0$ be such that $D(z_1,\varepsilon) \subset D(z_0,R_1)$.

(Refer Slide Time: 20:32)



Let $z \in D(z_1,\varepsilon)$,

$$|a_n\left((z-z_0)^{n-1}+(z-z_0)^{n-2}(z_1-z_0)+\cdots+(z_1-z_0)^{n-1}\right)| \le |a_n n R_1^{n-1}|$$

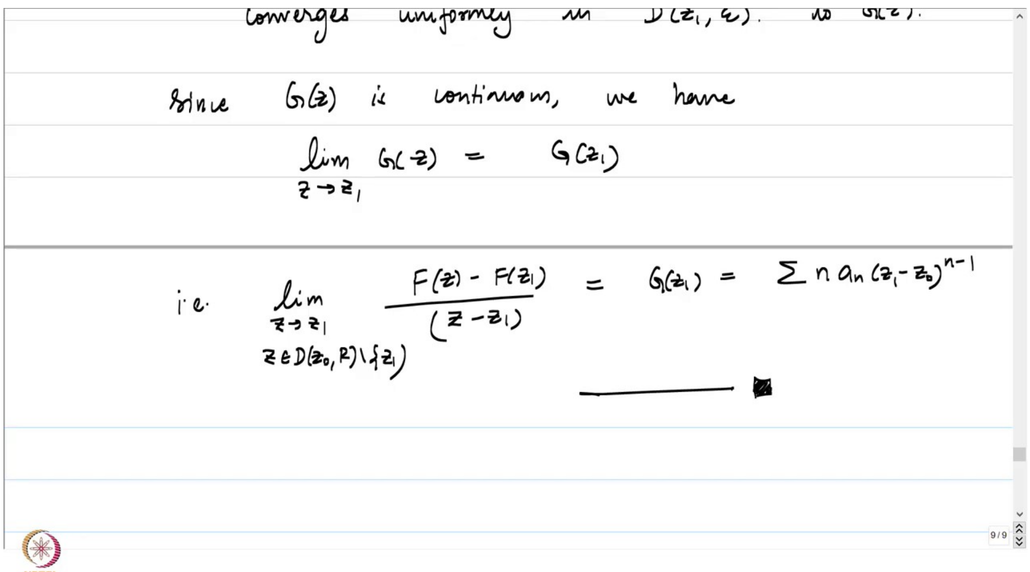
.

Since $\liminf_{n\to\infty}|a_n|^{-1/n} = R$ and $R_2 < R$, we have $N \in \mathbb{N}$ such that $\forall n > N$,
$|a_n|^{-1/n} > R_2 \implies |a_n| < \dfrac{1}{R_2^n}$
$\implies |a_n\left((z-z_0)^{n-1} + (z-z_0)^{n-2}(z_1-z_0) + \cdots + (z_1-z_0)^{n-1}\right)| \le n\left(\dfrac{R_1}{R_2}\right)^n$.
Hence by Weierstrass M-test, $\sum a_n\left((z-z_0)^{n-1} + (z-z_0)^{n-2}(z_1-z_0) + \cdots + (z_1-z_0)^{n-1}\right)$ converges uniformly in $D(z,\varepsilon)$ to $G(z)$. Since $G(z)$ is continuous, we have

$$\lim_{z\to z_1} G(z) = G(z_1)$$

.

(Refer Slide Time: 25:59)



That is,

$$\lim_{\substack{z\to z_1\\ z\in D(z_0,R)\setminus\{z_1\}}} \frac{F(z)-F(z_1)}{(z-z_1)} = G(z_1) = \sum_{n=1}^{\infty} n a_n (z_1 - z_0)^{n-1}$$

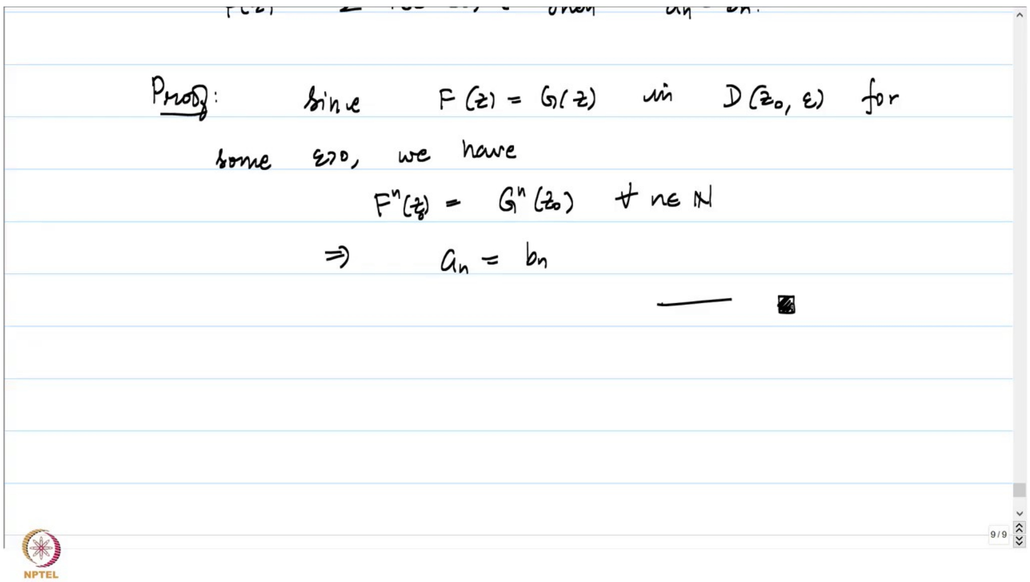
. Thus $F(z) = \sum a_n (z-z_0)^n$ is holomorphic on $D(z_0,R)$.

□

COROLLARY 2. *Let $F(z) = \sum a_n(z-z_0)^n$ on $D(z_0,R)$, where R is the radius of convergence of the power series. Then $a_n = \dfrac{F^n(z_0)}{n!}$.*

PROOF. The reader should verify that $F^m(z) = \sum_{n=m}^{\infty} n(n-1)\dots(n-m+1)a_n(z-z_0)^{n-m}$. Now, $F(z_0) = m(m-1)\dots 2\cdot 1 a_m = m!a_m \implies a_m = \dfrac{F^m(z_0)}{m!}$ □

(Refer Slide Time: 32:24)



COROLLARY 3. *If $G(z)$ is a power series $\sum b_n(z-z_0)^n$ such that $G(z) = F(z)$ in a neighborhood of z_0, where $F(z) = \sum a_n(z-z_0)^n$. Then $a_n = b_n$.*

PROOF. Since $F(z) = G(z)$ in $D(z_0, \varepsilon)$ for some $\varepsilon > 0$, we have $F^n(z) = G^n(z)\forall n \in \mathbb{N} \implies a_n = b_n$ □

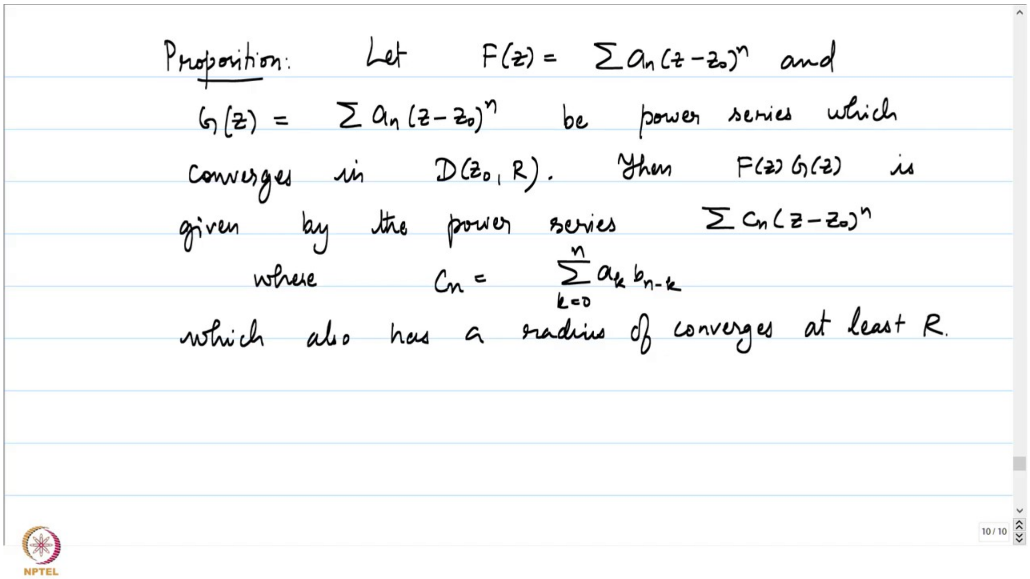
Now lets calculate the derivative of one of the important power series we discussed earlier, the exponential function.

Consider $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$. Then, $\dfrac{d}{dz}e^z = \sum_{n=1}^{\infty} \dfrac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = e^z$.

Now consider, $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$. Then, $\dfrac{d}{dz}\sin(z) = \dfrac{e^{iz} + e^{-iz}}{2} = \cos(z)$.

Similarly, $\dfrac{d}{dz}\cos(z) = -\sin(z)$.

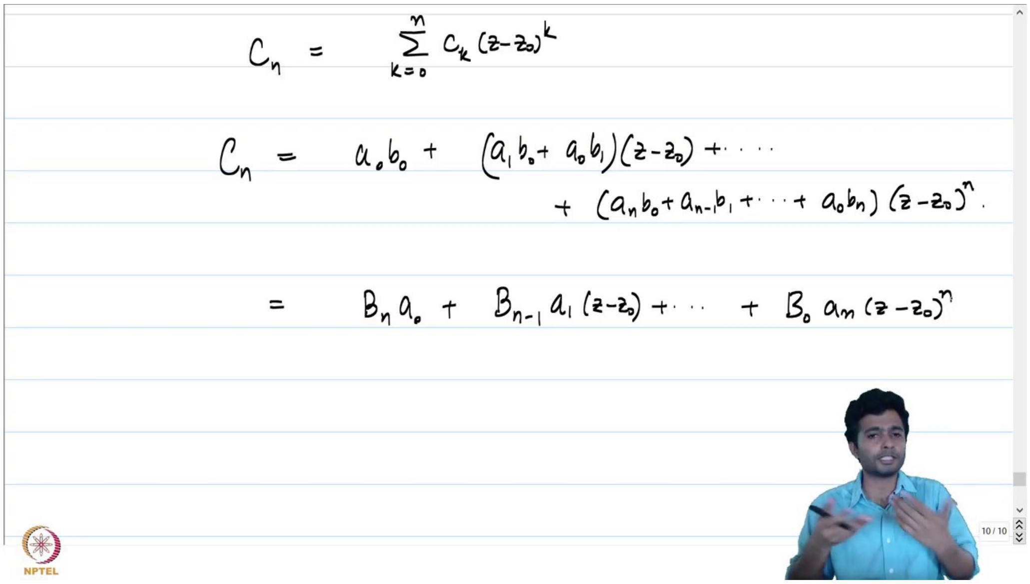
(Refer Slide Time: 36:39)



PROPOSITION 4. *Let* $F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ *and* $G(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$ *be power series which converges in* $D(z_0, R)$*. Then* $F(z)G(z)$ *is given by power series* $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ *where* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ *which also has a radius of convergence at least* R*.*

PROOF. Notice that for $z \in D(z_0, R)$, $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ converges absolutely. Let $A_n = \sum_{k=0}^{n} a_k (z - z_0)^k, B_n = \sum_{k=0}^{n} b_k (z - z_0)^k C_n = \sum_{k=0}^{n} c_k (z - z_0)^k$ and $D_n = B_n - G(z)$.

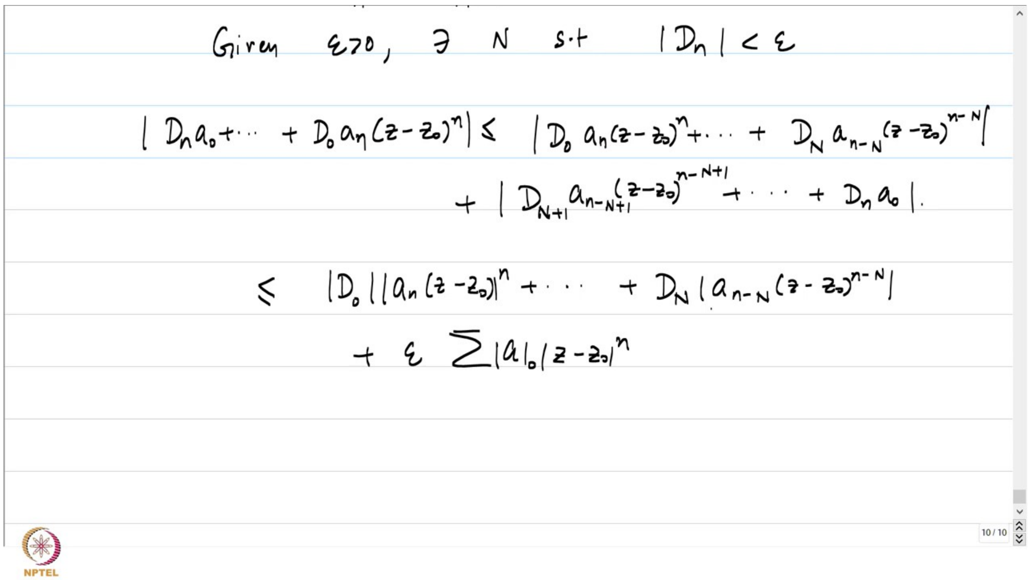
(Refer Slide Time: 41:39)



Then

$$
\begin{aligned}
C_n &= a_0 b_0 + (a_1 b_0 + a_0 b_1)(z - z_0) + \cdots + (a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n)(z - z_0)^n \\
&= B_n a_0 + B_{n-1} a_1 (z - z_0) + B_{n-2} a_2 (z - z_0)^2 + \cdots + B_0 a_n (z - z_0)^n. \\
&= G(z)\,(a_0 + a_1(z - z_0) + \cdots + a_n(z - z_0))^n + (D_n a_0 + D_{n-1} a_1 (z - z_0) + \cdots + D_0 a_n (z - z_0)^n) \\
C_n &= G(z) A_n + (D_n a_0 + D_{n-1} a_1 (z - z_0) + \cdots + D_0 a_n (z - z_0)^n)
\end{aligned}
$$

(Refer Slide Time: 45:33)



Consider $D_n a_0 + D_{n-1} a_1 (z - z_0) + \cdots + D_0 a_n (z - z_0)^n$. For a given $\varepsilon > 0, \exists N$ such that $|D_n| < \varepsilon \, \forall n \geq N$.

$$|D_n a_0 + \cdots + D_0 a_n (z - z_0)^n| \leq |D_0 a_n (z - z_0)^n + \cdots + D_N a_{n-N} (z - z_0)^{n-N}|$$

$$+ |D_{N+1} a_{n-N+1} (z - z_0)^{n-N+1} + \cdots + D_n a_0|$$

$$\leq |D_0||a_n (z - z_0)|^n + \cdots + |D_N||a_{n-N} (z - z_0)^{n-N}|$$

$$+ \varepsilon \sum_{n=0}^{\infty} |a_n| (z - z_0)|^n$$

$$|D_n a_0 + \cdots + D_0 a_n (z - z_0)^n| \leq |D_0||a_n (z - z_0)|^n + \cdots + |D_N||a_{n-N}||z - z_0|^{n-N} + \varepsilon \alpha$$

$$\lim_{n \to \infty} |D_n a_0 + \cdots + D_0 a_n (z - z_0)^n| \leq \varepsilon \alpha$$

$$\limsup_{n \to \infty} |D_n a_0 + \cdots + D_0 a_n (z - z_0)^n| = 0$$

$$\Rightarrow \lim_{n \to \infty} |D_n a_0 + \cdots + D_0 a_n (z - z_0)^n| = 0$$

$$\Rightarrow \lim_{n \to \infty} C_n = F(z) G(z).$$

□