Complex Analysis
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Lecture – 1
Field of Complex Numbers

Welcome to this course on complex analysis. This is a first course on complex analysis. The prerequisites that will be needed for this course would be sound understanding of basic linear algebra and basic real analysis. If you have seen some abstract algebra that will certainly help, but other than the first lecture the rest of the cause will not use much material or much knowledge from abstract algebra.

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Regarding the textbook, so let me note down a few books. I should warn that we do not have a prescribed textbook for this course; however, most of the material that will be covered in this course can be found in almost all the classical books written in complex analysis. However, I would like to refer the following books for this course. The first one being Complex Analysis by Stein and Shakarchi.

The second is titled "Functions of One Complex Variable". There are 2 parts in this book, so the material that we will cover in this course will be found in the first part and it is by John B. Conway. Of course, there are many, many, many beautiful books written on this subject, maybe I should give a few other references.

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Other references

* Complex Analysis by Lars Ahlfors.
* Complex Analysis by Theodore Gamelin.
* Real & Complex analysis by Walter Rudin.

There is this age old classic it is called “Complex Analysis” by Lars Ahlfors. There is also “Complex Analysis” by Theodore Gamelin. There is also this fantastic book “Real and Complex Analysis” with a slightly different approach by Walter Rudin. Of course, there are many more books, let me not write down all of them, but I would also suggest that you refer to these other references maybe a second or a third reading. Okay that is about textbooks.

There will be weekly assignments in this course and you are really strongly encouraged to work on these problems on your own, you should spend some time sitting and thinking about these problems that will give you much better clarity on the subject material that will be covered. More or less everything introduced the course to you, so let us now begin the study of the subject. In a course in real analysis you would have started by studying rational numbers.

You would have seen that rational numbers have certain deficiencies. For example if you look at some Cauchy sequences in rational numbers they do not converge and real numbers were constructed precisely to address this particular problem. Real numbers are the complete field which contain the rational numbers and it is unique up to some field isomorphisms.

You would have developed and studied an entire rich and beautiful theory of real analysis on this field of real numbers, but then from an algebraic point of view, real numbers also have certain drawbacks. So there are polynomials, for example, which do not have roots in the real numbers. For example, $x^2 + 1$ is a polynomial which does not have a root in the field of real numbers. Complex numbers was historically constructed in order to address this particular problem. So let us start this course by recalling what a field is.

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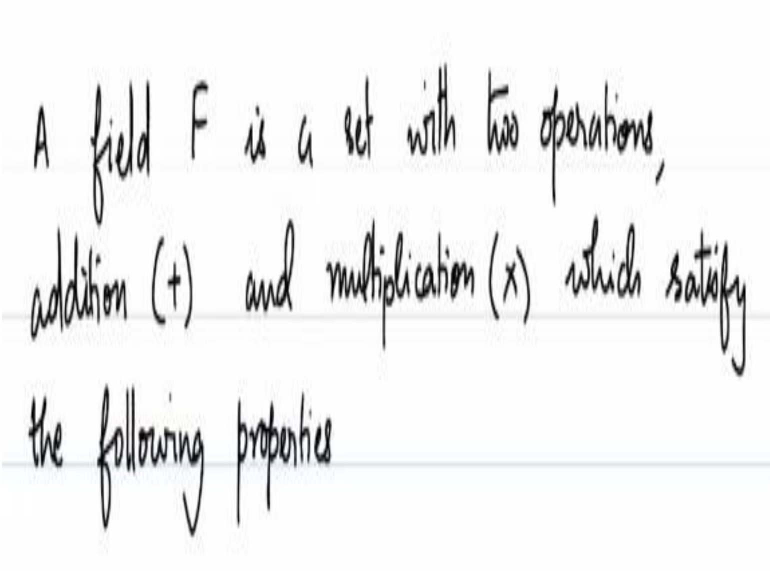

So, a field F is a set with 2 operations. Let us call these operations addition which are denote by (+) and multiplication which is being denoted by (×), which satisfy the following properties.

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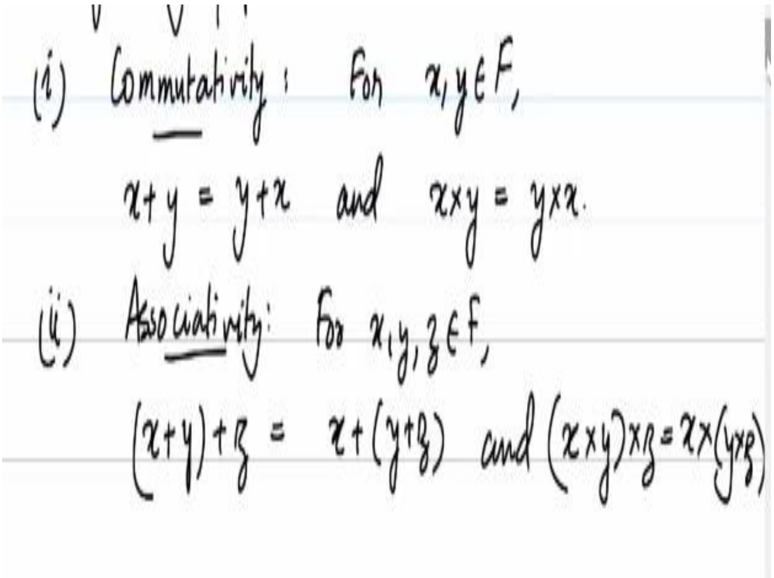

First one is commutativity. If you take 2 elements $x, y \in$ F, $x + y = y + x$. The order in which we take the sum does not matter and $x \times y = y \times x$, the order in which you multiply also does not matter. Second one is associativity. For $x, y, z \in$ F,$(x + y) + z = x + (y + z)$. Similarly,$(x \times y) \times z = x \times (y \times z)$. Associativity is a property which holds for both addition and multiplication.

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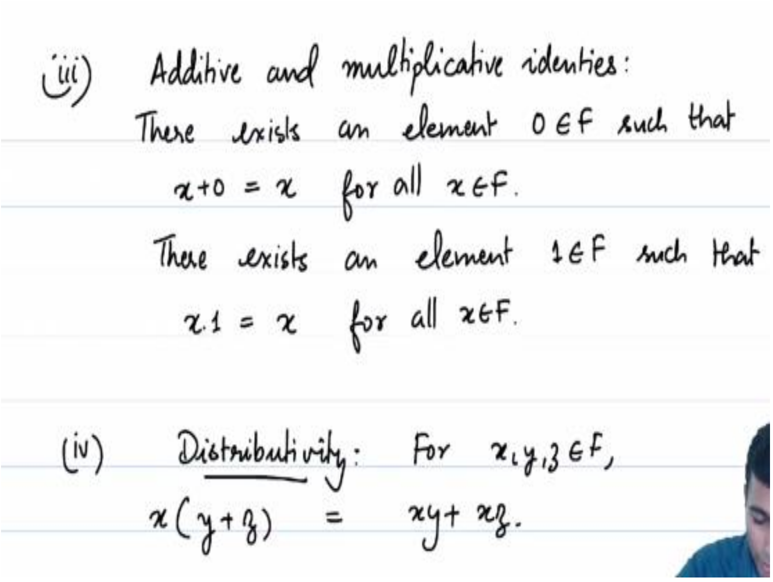

The third property is existence of additive and multiplicative identities. There exists an element 0ϵ F such that $x + 0 = x$ for all $x \epsilon$ F. This is the additive identity in F. There exists an element 1ϵ F such that $x \times 1 = x$ for all $x \epsilon$ F. So basically, there are two special elements in the field F.

Fourth property is distributivity. The addition and the multiplication operations, they interact with each other. For $x, y, z \epsilon$ F, $x \times (y + z) = x \times y + x \times z$.

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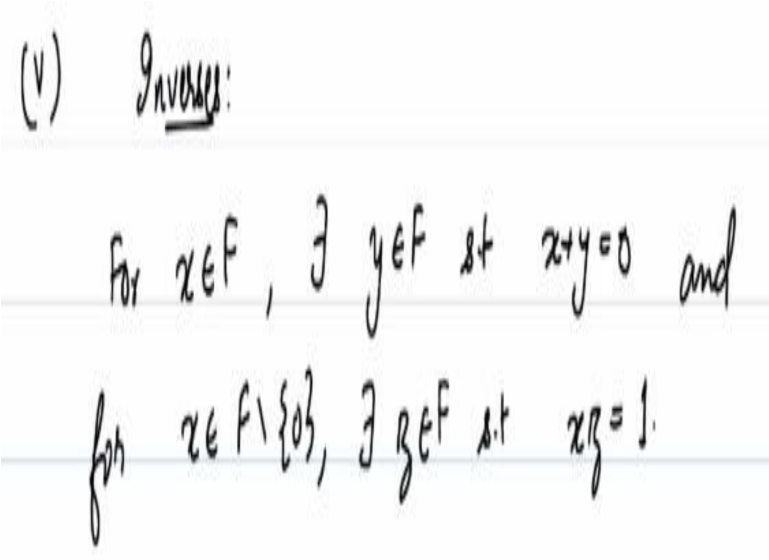

Finally, existence of inverses. For $x \epsilon$ F, there exists $y \epsilon$ F such that $x + y$ gives you the additive identity 0 and for $x \epsilon$ F\{0}, there exists $z \epsilon$ F such that $x \times z = 1$. The multiplicative inverse exists only for nonzero elements in the field F. This is the set of all properties which needs to be satisfied by the two operations for our given set to be a field. So that is the definition of a field and we are familiar with the field of real numbers.

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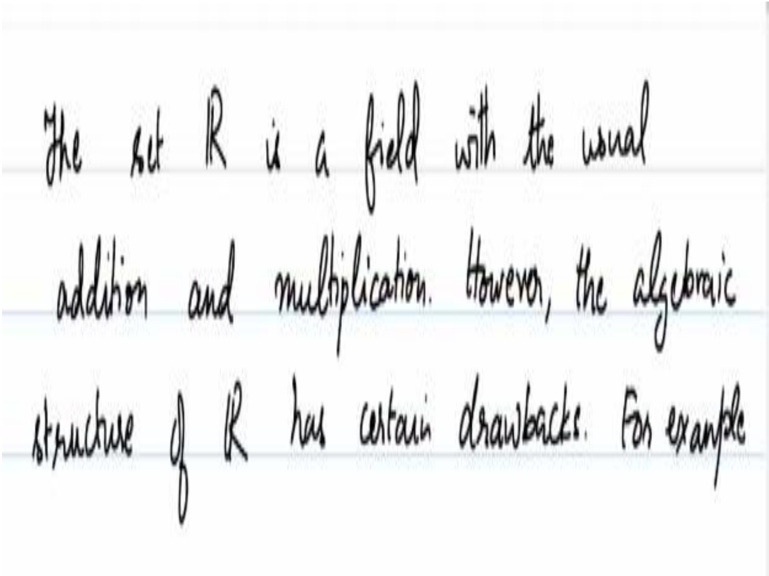

The set ℝ is a field with the usual addition and multiplication operation. However, the algebraic structure of ℝ has certain drawbacks. For example, all polynomials need not have roots in the field of real numbers.

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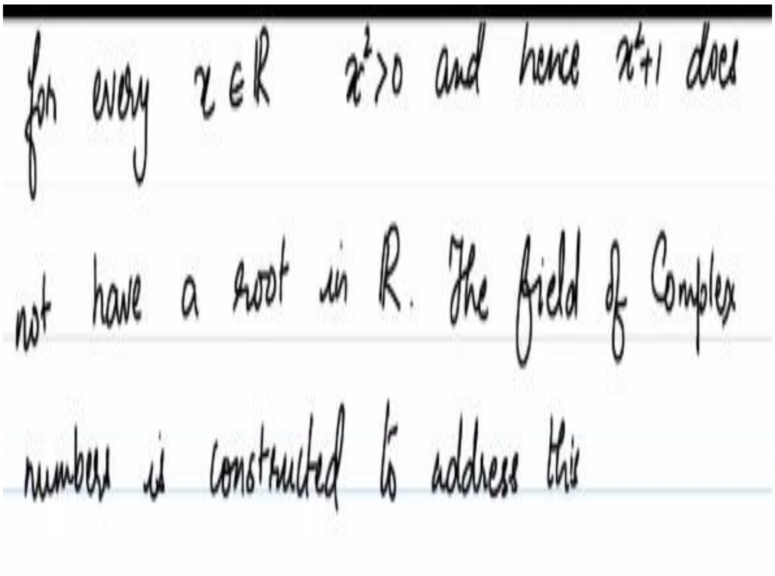

For every $x \epsilon \mathbb{R}$, $x^2 \geq 0$ and hence the polynomial $x^2 + 1$ does not have root in ℝ. Field of complex numbers is constructed to address this drawback. Let me just start with definition for a field of complex numbers.

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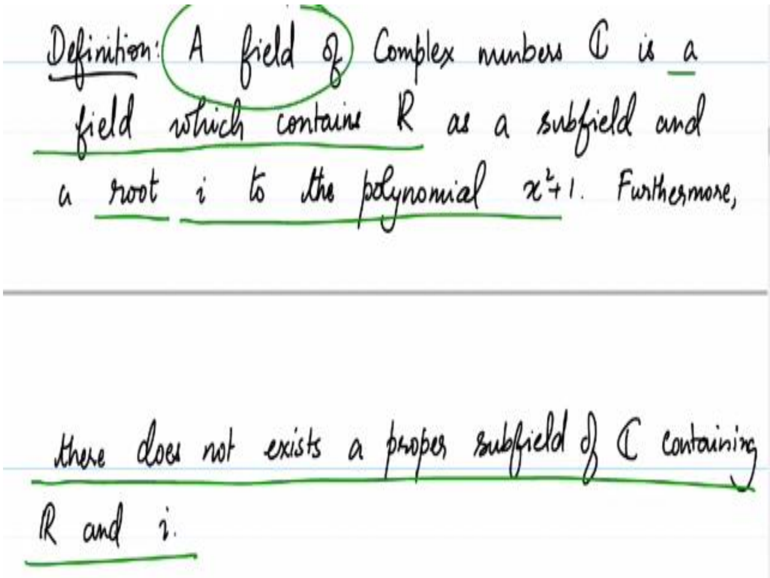

Definition: A field of complex numbers $\mathbb{C}$ is a field which contains $\mathbb{R}$ as subfield and a root i to the polynomial $x^2 + 1$. Furthermore, the field of complex numbers is the smallest such field which contains $\mathbb{R}$ and i.

Rephrasing it, there does not exist a proper subfield of $\mathbb{C}$ containing $\mathbb{R}$ and i. When put differently, this says that $\mathbb{C}$ is generated by $\mathbb{R}$ and i or if you have $\mathbb{C}'$, a subfield of $\mathbb{C}$ which contains $\mathbb{R}$ and i, then $\mathbb{C}'$ would necessarily be equal to $\mathbb{C}$.

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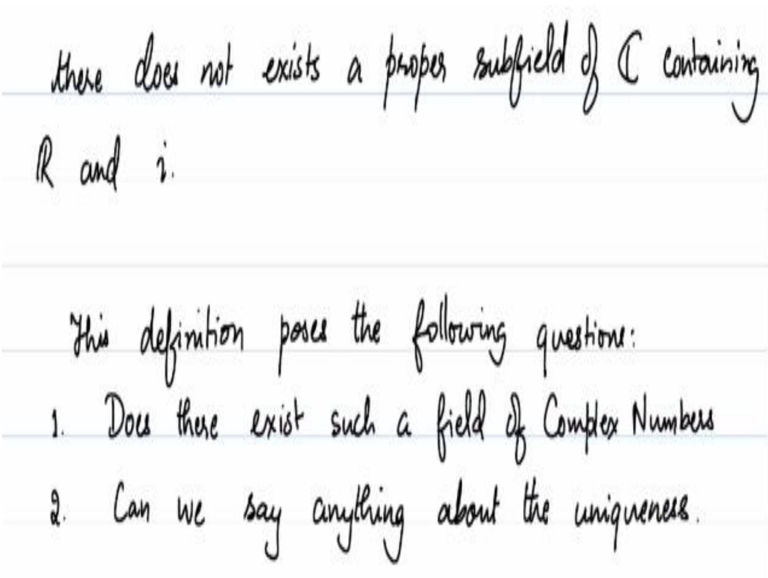

This definition poses the following questions. The first one being does there exist such a field of complex numbers and the second one is can we say anything about the uniqueness.

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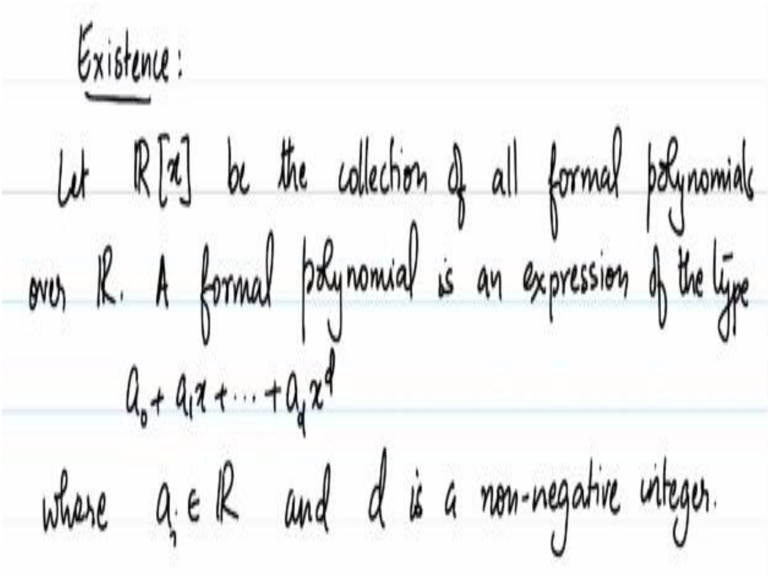

In order to address the existence of such a field of complex numbers, we will be using some notions from abstract algebra, more precisely from ring theory. If you have not seen a course on abstract algebra, there is no problem, you may skip the remaining part of the lectures.

You can assume the existence and the uniqueness of such a field of complex numbers, move over to the next lecture. From the next lecture onwards, there will be more focus on the analysis on such a field of complex numbers, you will not be needing too much of background in abstract algebra. However, in this lecture, we will be assuming some amount of knowledge in ring theory.

Let $\mathbb{R}[x]$ be the collection of all formal polynomials over $\mathbb{R}$, the field of real numbers. A formal polynomial is an expression of the type $a_0 + a_1x + \cdots + a_dx^d$, where a_i are real numbers and d is a non-negative integer.

We will be familiar with the addition of polynomials, multiplication of polynomials. Let me not give all those things as definitions again.

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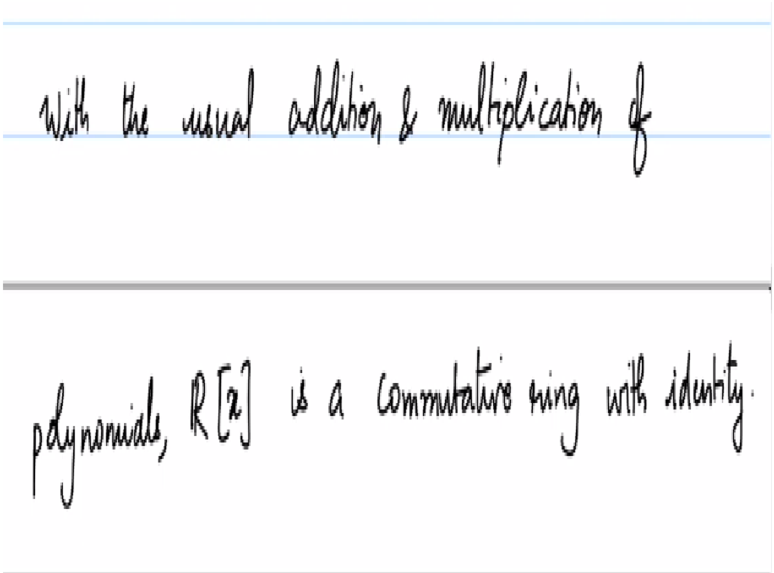

Let me just note that with the usual addition and multiplication of polynomials, $\mathbb{R}[x]$ is commutative ring with identity. So if you have not seen abstract algebra before, a commutative ring is set with two operations with almost all these properties satisfied except the existence of multiplicative inverse.

What are the identities here? The zero polynomial is the additive identity and the constant polynomial 1 will be the multiplicative identity. The first thing to note is that $\mathbb{R}[x]$ is a commutative ring which is not a field, not all elements can be inverted.

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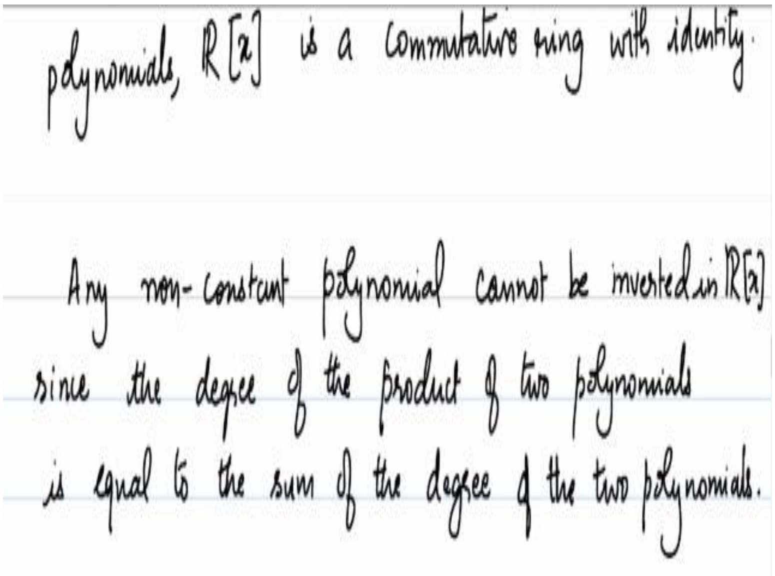

In fact any non-constant polynomial is not a unit. There does not exist a multiplicative inverse for any non-constant polynomial in $\mathbb{R}[x]$. Why is that the case? Because we have a notion of the degree of a polynomial, d and if you look at the product of two polynomials, the degree adds up. If you want an inverse for a polynomial $p(x)$, there should exist a $q(x)$ such that $p(x)q(x) = 1$.

The constant polynomial 1 has degree 0. However if $p(x)$ has degree greater than 0, then degree of $p(x)q(x)$ will always be greater than or equal to 1 and hence it cannot ever be a constant polynomial.

Since the degree of the product of two polynomials is equal to the sum of the degree of the polynomials and the degree is always a non-negative number. Therefore non-constant polynomial can never have an inverse.

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Let us take the polynomial $x^2 + 1$. Since $x^2 + 1$ is a non-constant polynomial, it is not invertible, the ideal generated by $x^2 + 1$, $\langle x^2 + 1\rangle$ will not be the entire ring.

If a is an element in the ideal, for any element b in the ring, ab will be in the ideal and the ideals are the right objects with which we take quotients in a ring.

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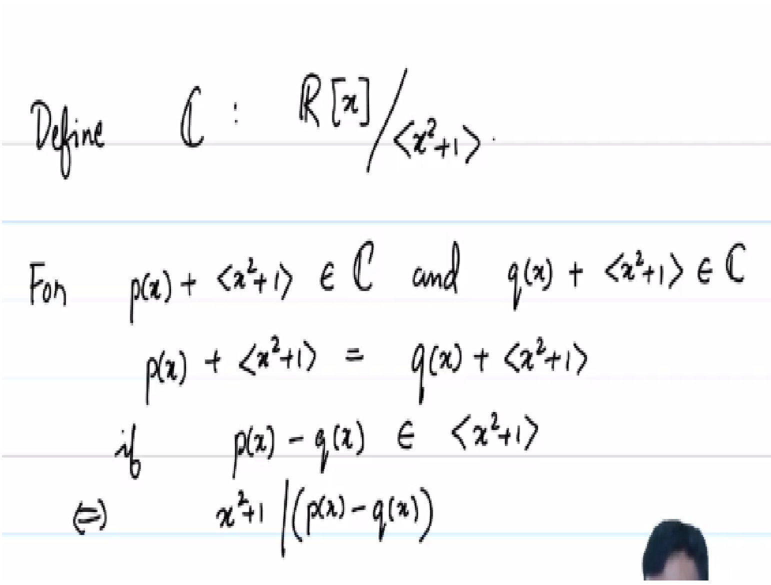

Let us define $\mathbb{C} := \mathbb{R}[x]/\langle x^2 + 1\rangle$.

The objects here will be cosets of the ideal $\langle x^2 + 1\rangle$. The operation in C are given by;

for the cosets $p(x) + \langle x^2 + 1\rangle \in \mathbb{C}$ and $q(x) + \langle x^2 + 1\rangle \in \mathbb{C}$, $p(x) + \langle x^2 + 1\rangle = q(x) + \langle x^2 + 1\rangle$ if $p(x) - q(x) \in \langle x^2 + 1\rangle$ if and only if $x^2 + 1$ being a factor of $p(x) - q(x)$. This is the equivalence relation of the cosets involved.

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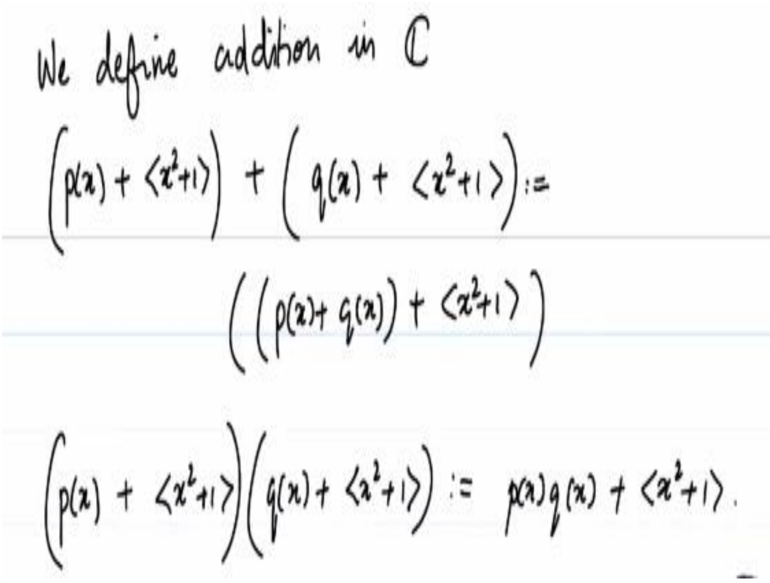

The sum of two elements in $\mathbb{C}$ is defined by, for $p(x)+\langle x^2+1\rangle, q(x)+\langle x^2+1\rangle \in \mathbb{C}$, $p(x)+\langle x^2+1\rangle + q(x)+\langle x^2+1\rangle = (p(x)+q(x))+\langle x^2+1\rangle$ and the multiplication of two elements in C is defined as, for $p(x)+\langle x^2+1\rangle, q(x)+\langle x^2+1\rangle \in \mathbb{C}$,

$$(p(x)+\langle x^2+1\rangle)\times(q(x)+\langle x^2+1\rangle)=(p(x)q(x))+\langle x^2+1\rangle$$

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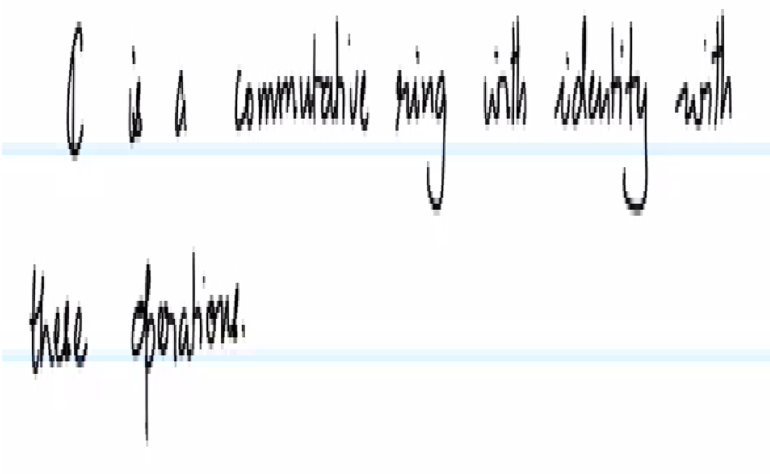

Let me invoke the knowledge from ring theory to say that $\mathbb{C}$ is a commutative ring with identity with these operations. So we have found a commutative ring which is a candidate. So why is this a candidate?

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Define $\varphi : \mathbb{R} \to \mathbb{C}$ given by

$\varphi(a) := a + \langle x^2+1 \rangle$

Claim: φ is injective

Proof: If $\varphi(a) = \varphi(b)$, then

$a + \langle x^2+1 \rangle = b + \langle x^2+1 \rangle$.

$\Rightarrow (x^2+1) \mid (a-b)$.

Since $\deg(a-b) = 0$, we have

$(a-b) = (x^2+1).0 = 0$

Define $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ given by, for $a \epsilon \mathbb{R}$, $\varphi(a) := a + \langle x^2 + 1 \rangle$. So easy to see that this ring homomorphism.

Claim: φ is injective, φ embeds $\mathbb{R}$ into $\mathbb{C}$.

To check it is injective what do we have to do?

Proof of the claim,

$\varphi(a) = \varphi(b) \Rightarrow a + \langle x^2 + 1 \rangle = b + \langle x^2 + 1 \rangle \Rightarrow x^2 + 1$ divides $a - b$, but $a - b$ is a real number. Any divisible, any factor or any multiple of $x^2 + 1$ should have at least degree 2 or else it should be the 0 polynomial. Since degree of $a - b = 0$. We have $a - b = (x^2 + 1) \times 0 = 0 \Rightarrow a = b$.

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$\Rightarrow a = b.$ — ∎

Let $i := x + \langle x^2+1 \rangle$ in $\mathbb{C}$

Then $(i^2+1) = (x + \langle x^2+1 \rangle)^2 + (1 + \langle x^2+1 \rangle)$

$= (x^2+1) + \langle x^2+1 \rangle$

$= 0 + \langle x^2+1 \rangle$

$\Rightarrow i^2+1 = 0$ in $\mathbb{C}$

Immediately we see that the map φ from $\mathbb{R}$ to $\mathbb{C}$ is an embedding, it actually gives us a copy of $\mathbb{R}$ in the commutative ring $\mathbb{C}$. That is one aspect of the definition solved and like it may let us now that we can get hold of a root of $x^2 + 1$ in this particular commutative ring. So let i be defined to be, $i \coloneqq x + \langle x^2 + 1 \rangle$ in $\mathbb{C}$.

Recall that $\mathbb{C}$ is $\mathbb{R}[x]/\langle x^2 + 1 \rangle$, so you look at the coset represented by x, then $i^2 + 1 = (x + \langle x^2 + 1 \rangle)^2 + (1 + \langle x^2 + 1 \rangle)$ in $\mathbb{C}$. By the definitions of multiplication and addition that we have defined in the quotient, $i^2 + 1 = (x^2 + 1) + \langle x^2 + 1 \rangle$ = 0 + $\langle x^2 + 1 \rangle \Rightarrow i^2 + 1 = 0$ in $\mathbb{C}$.

Hence, we have both solution to $x^2 + 1$ = 0 and a copy of $\mathbb{R}$ sitting inside $\mathbb{C}$. So we are at least two of the aspects in the definition of field of complex numbers.

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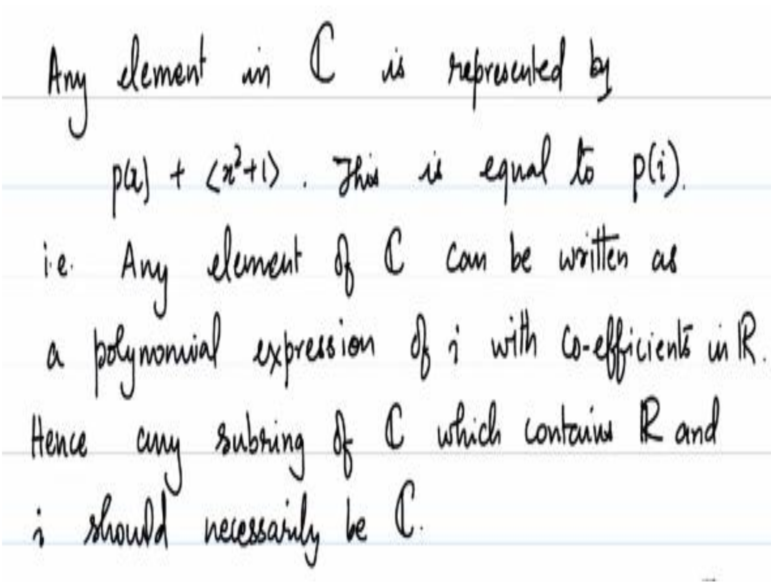

In fact, we also can say something more about the characterization, namely that this is actually generated by $\mathbb{R}$ and i and we say that any element, you look to any element in $\mathbb{C}$, this is represented by $p(x) + \langle x^2 + 1 \rangle$, for some polynomial $p(x)$. Then this is equal to $p(i)$.

That is, any element of $\mathbb{C}$ can be written as polynomial expression of i with coefficients in $\mathbb{R}$. But what can we say about any field of $\mathbb{C}$ which contains $\mathbb{R}$ and i? It will certainly have any polynomial expression in i with coefficients because it is a field that is closed under both multiplication and addition. Hence any subring of $\mathbb{C}$ which contains $\mathbb{R}$ and i should necessarily be equal to $\mathbb{C}$.

The only thing that is to be checked is whether it is a field. As of now whatever we have constructed $\mathbb{C}$, it is just a commutative ring with the identity.

If we establish that it is indeed a field as well, we would have proved the existence of field of complex numbers.

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The polynomial x^2+1 is irreducible in $\mathbb{R}[x]$.

x^2+1 is hence a prime element.

Let $(x^2+1) \mid p(x)q(x)$

If $x^2+1 \nmid p(x)$, then since x^2+1 is irreducible,

$\exists$ polynomials $\alpha(x)$ and $\beta(x)$ s.t.

$$\alpha(x)(x^2+1) + \beta(x)p(x) = 1.$$

So let us now work towards proving that this particular commutative ring is indeed a field, but to do that, let us focus on the polynomial x^2+1 in $\mathbb{R}[x]$. The polynomial x^2+1 is an irreducible element in the commutative ring $\mathbb{R}[x]$. So the first observation would be that an irreducible element in $\mathbb{R}[x]$ would be prime, more specifically x^2+1 is hence prime element. (Recall what was a prime element was: in a ring A, p is defined to be a prime element if whenever p divides ab, the product of two elements a and b in ring, p either divides a or p divides b.)

So, the claim here is that x^2+1 is a prime element in $\mathbb{R}[x]$. Let us just check that very quickly. Let x^2+1 divide $p(x)q(x)$, let us prove that it divides one of them. So without loss of generality if x^2+1 divides $p(x)$, then we have already proved.

If x^2+1 does not divide $p(x)$, x^2+1 is an irreducible polynomial and therefore if you look at the greatest common divisor of x^2+1 and $p(x)$, it should necessarily be 1. So it will be a unit, so it will be 1 here. Let me rephrase it this way since x^2+1 is irreducible, there exist polynomials $\alpha(x)$ and $\beta(x)$ such that $\alpha(x)(x^2+1)+\beta(x)p(x)=1$.

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$\exists$ polynomials $\alpha(x)$ and $\beta(x)$ s.t.

$$\alpha(x)(x^2+1) + \beta(x)p(x) = 1.$$

Multiplying $q(x)$ to this equation, we have

$$\underline{\alpha(x)(x^2+1)q(x)} + \underline{\beta(x)p(x)q(x)} = q(x)$$

Multiplying $q(x)$ to the equation above, we have $\alpha(x)(x^2+1)q(x) + \beta(x)p(x)q(x) = q(x)$, but if you focus on the left hand side, the first term contain the factor of x^2+1 and second term contain the product $p(x)q(x)$ and hence x^2+1 divides both terms of left hand side, therefore x^2+1 divides $q(x)$.

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$$\alpha(x)(x^2+1)q(x) + \beta(x)p(x)q(x) = q(x)$$

Since L.H.S is divisible by x^2+1, we have

$$x^2+1 \mid q(x).$$

Since $\langle x^2+1 \rangle$ is a prime ideal, we have

$\mathbb{C}$ is an integral domain.

Then x^2+1 is a prime element and therefore the ideal, $\langle x^2+1 \rangle$ is a prime ideal. We know that when we go modulo a prime ideal, we get an integral domain. Since $\langle x^2+1 \rangle$ is a prime ideal, we have $\mathbb{C}$ is an integral domain.

The fact that $\mathbb{C}$ is a field is now established only after showing that every nonzero element can be inverted.

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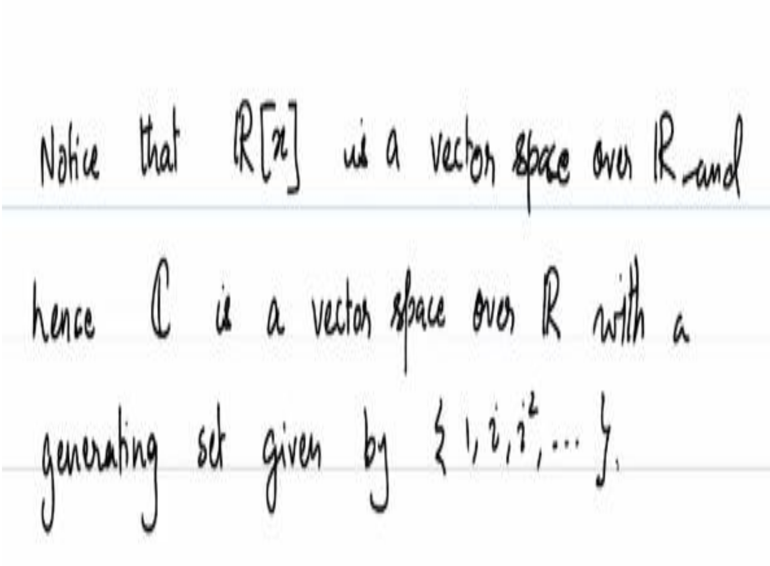

Notice that $\mathbb{R}[x]$, because there is a copy of $\mathbb{R}$ is in $\mathbb{R}[x]$, is a vector space over $\mathbb{R}$ with generating set given by $\{1, x, x^2, \dots\}$ and since $\mathbb{C}$ is a quotient of our $\mathbb{R}[x]$ by an ideal, $\mathbb{C}$ is also a vector space over $\mathbb{R}$ with generating set given by $\{1, i, i^2, \dots\}$.

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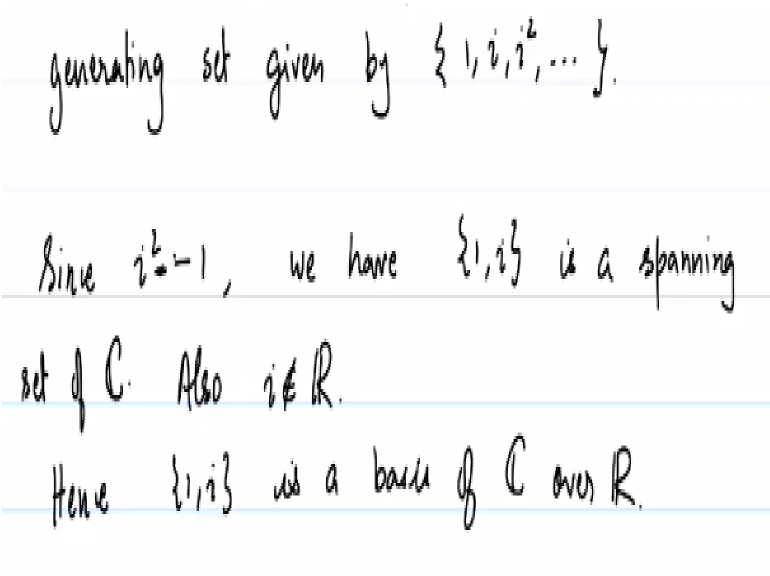

Since $i^2 = -1$, we have $\{1, i\}$ is a spanning set of $\mathbb{C}$. So we have a set consisting of 2 elements, which is a spanning set of $\mathbb{C}$. Also note this that i does not belong to $\mathbb{R}$. Why is that? It is because if i belongs to $\mathbb{R}$, then there is a real number a in $\mathbb{R}$ such that $x + \langle x^2+1 \rangle = a + \langle x^2+1 \rangle \Rightarrow x - a \in \langle x^2+1 \rangle$ which is not possible since $x - a$ will have degree one.

So, i does not belong to $\mathbb{R}$ and therefore 1 and i turn out to be linearly independent. So hence $\{1, i\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$.

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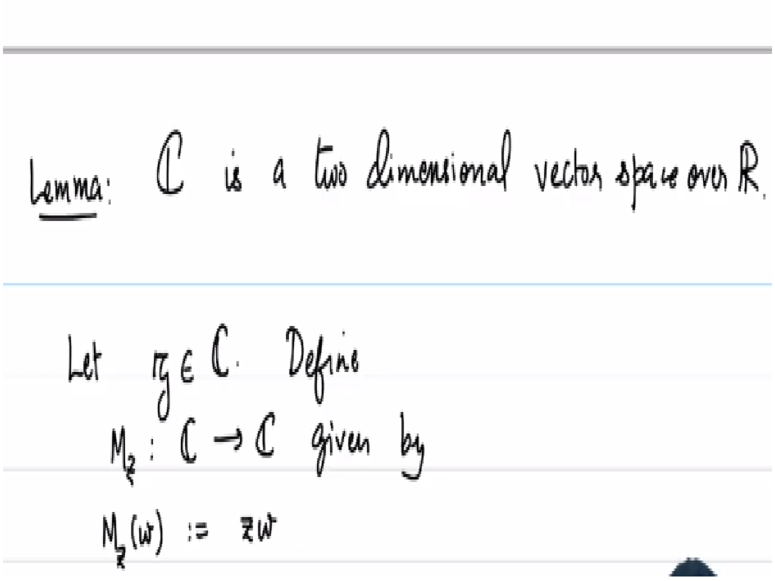

We have now just established a lemma.

Lemma: ℂ is a two-dimensional vector space over ℝ.

We have still not proved that ℂ is a field, so in order to do that let us take a non-zero arbitrary element z in ℂ. Define M_z: ℂ → ℂ given by,$M_z(w) := zw$. We have already checked that ℂ is an integral domain, this is left multiplication by z and you should check that M_z is actually an ℝ linear map, it is a linear transformation.

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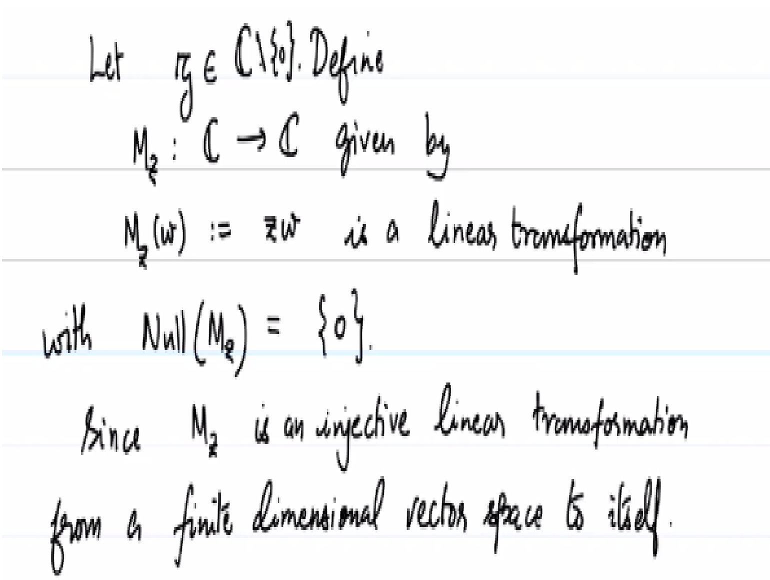

This is an easy check and you will immediately note that the null space of M_z is $\{0\}$, since null space contain those w such that $zw = 0$, but we already checked that ℂ is an integral domain and this can happen only if $w = 0$, as $z \neq 0$.

Therefore, M_z is an injective linear transformation from a two-dimensional vector space to itself. We know that an injective linear transformation from a finite dimensional vector space to itself should necessarily be surjective by the rank-nullity theorem.

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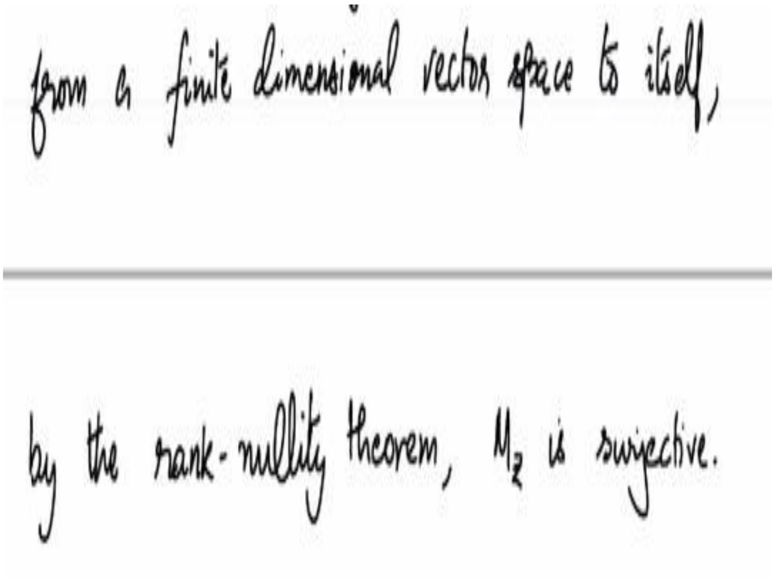

That means there exists some w' such that $zw' = 1$. Hence z is invertible. Therefore $\mathbb{C}$ is a field.

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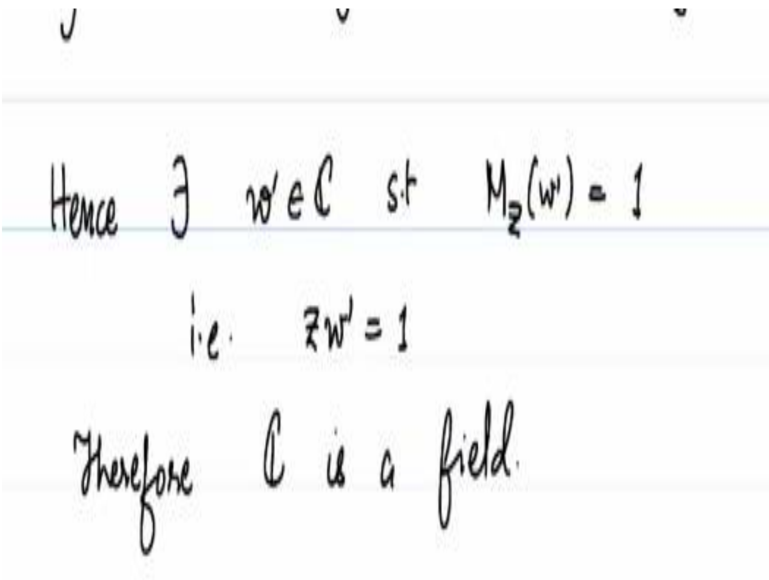

We have established every aspect of $\mathbb{C}$ being a field of complex numbers. Now have one such field of complex numbers. We have answered one of the questions that arose from the definitions satisfactorily.

Let us now move over to the second one, that is more easier than this, the question of uniqueness.

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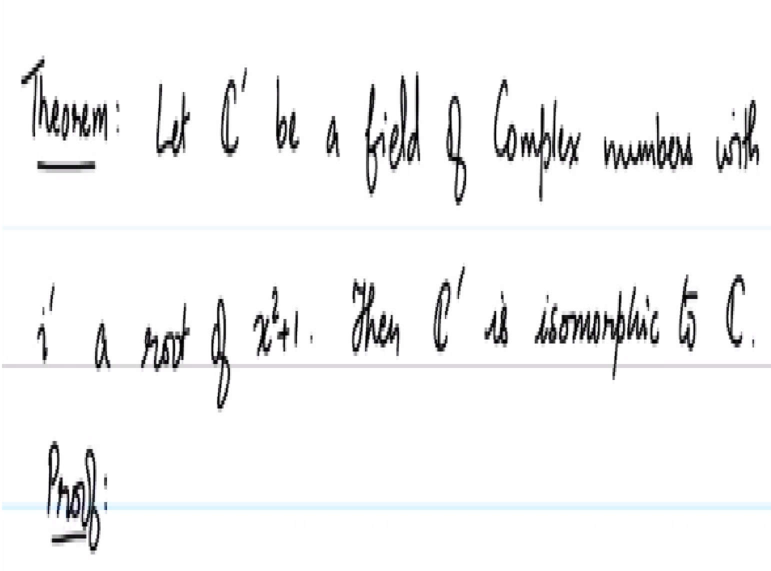

Theorem: Let $\mathbb{C}'$ be a field of complex numbers with i' a root of $x^2 + 1$. Then $\mathbb{C}'$ is isomorphic to $\mathbb{C}$.

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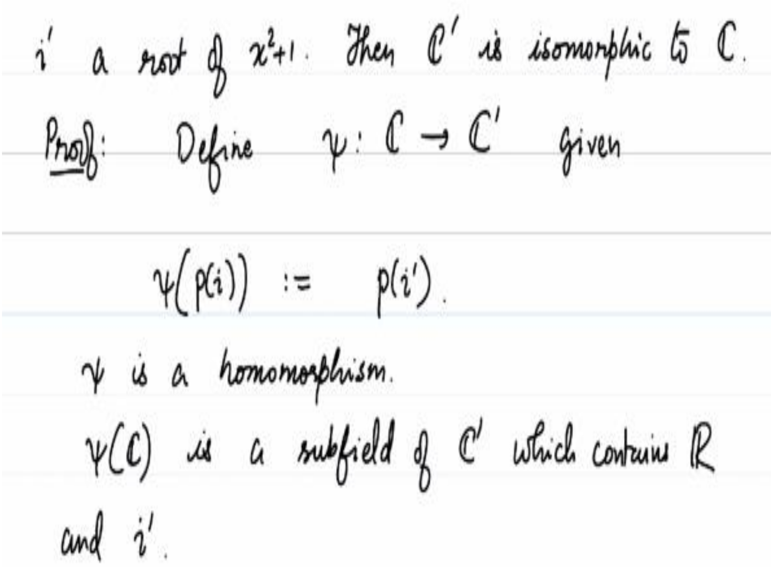

Proof: Define $\psi : \mathbb{C} \to \mathbb{C}'$ given by for $p(i) \epsilon \mathbb{C}$, $\psi(p(i)) = p(i')$. So notice that it will be a field homomorphism from $\mathbb{C}$ to $\mathbb{C}'$.

$\psi(\mathbb{C})$ is a subfield of $\mathbb{C}'$, but this $\psi(\mathbb{C})$ has certain characteristics , it contains $\mathbb{R}$ because its identity on $\mathbb{R}$ it will send $\mathbb{R}$ to $\mathbb{R}$. It also contains i' because $\psi(i) = i'$ by the very definition of ψ.

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Since $\mathbb{C}'$ is a field of complex numbers, we have

$$\psi(\mathbb{C}) = \mathbb{C}'$$

Hence ψ is a field isomorphism.

Since ℂ′ is field of complex numbers, this forces $\psi(\mathbb{C})$ to be entire $\mathbb{C}'$ and hence ψ is a field isomorphism. So we have established remarkable aspect here. It tells us that any two field that satisfy the properties in the definition of complex number, they should necessarily be isomorphic to each other and therefore we could study analysis over any one such field that we can construct. In the next lecture, we will get hold of another such construction of a field of complex numbers and that will be more handy to work with and we will be doing analysis on that field of complex numbers some more.