Measure Theory
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Module No # 02
Lecture No # 09
Uniqueness of Elementary Measure and Jordan Measurability – Part 2

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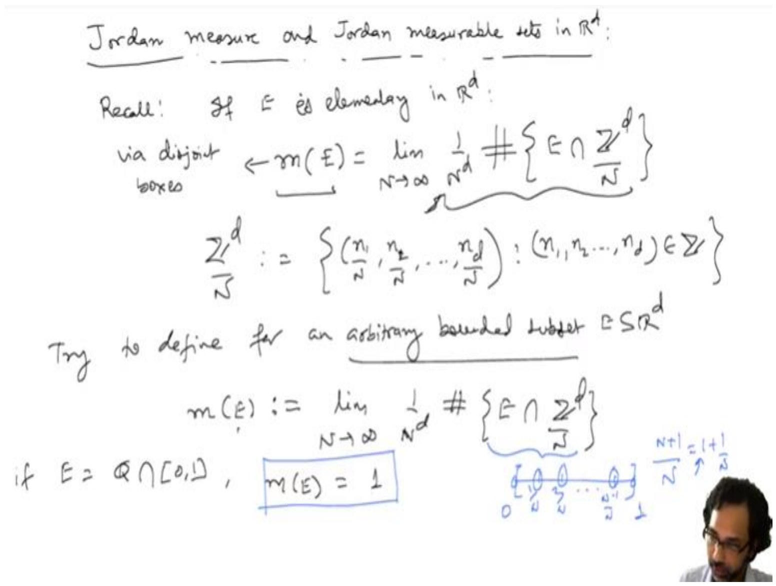

Now the next part which is Jordan measure and Jordan measureable sets in Rd so we have defined an elementary measure for elementary sets in Rd. Now we would like to increase collection of sets where we define our measure one choice remember that we call that if E is elementary in Rd. Then we had this formula m E was the limit as N goes to infinity 1 over N to the power d the count of the set E intersection with Z to the power d over N, where Z to the power d over N was the set n1 / N, n2 / N, nd / N such that n1, n2 up to nd is the (()) (01:50) in of integers.

So we know this formula and now one could try to define for an arbitrary bounded subset E of Rd m of E. Now this is the definition remember that our definition of m E was via disjoint boxes and you took the sum of disjoint the measure of this boxes. But we prove that this is equal to this limit this is some kind of counting but by normalizing it by a relevant factor N to the power d.

We get some we get formula which gives you m E the measure of elementary measure of E but now I am going to define this measure for any arbitrary bounded subset E to by this limiting formula. So this is our (()) (03:20) because for any arbitrary bounded subset we do not have an immediate volume formula by just taking the product of sides of the intervals. So we can try to definite it using this limit formula.

So but now notice that is E is the subset of rational's inside the interval 0, 1 then the measure of E via this formula is simply going to be 1. So this is for d = 1 now so we have this interval 0 to 1 and for any fixed n you are dividing this interval into chunks of length 1 / n. And so these points 1 over n, 2 over n up to n -1 over n belong to this set E intersection Z over n and so you will have how many points?

1 to up to n – 1 but we will also have to consider these 2 points so you will have in total N+1 points and then you divide by N. So you will just have 1 + 1 over N and when you take the limit as N goes to infinity you will just get 1. So we will have the measure of this set E which is the rational's n 0, 1 is simply 1.

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But if $y \in \mathbb{R} \setminus \mathbb{Q}$ say $y = \sqrt{5}$

$m(E+\sqrt{5}) = 0$ because $\underbrace{(E+\sqrt{5})}_{\mathbb{Q}\cap[0,1]} \cap \frac{\mathbb{Z}}{N} = \phi$ for any N.

$$m(E) = \lim_{N\to\infty} \frac{1}{N^d} \# \left\{ E \cap \frac{\mathbb{Z}^d}{N} \right\}$$

$0 \le \frac{p}{q} \le 1$

$\frac{p}{q} + \sqrt{5} = \frac{k}{N}$

fails the translation-invariance property.

$\Rightarrow \sqrt{5} = \frac{k}{N} - \frac{p}{q} \in \mathbb{Q}$

(a contradiction)

But if y is irrational number so let us say y = square root of 5 then if you translate this set E / square root of 5. Then the measure of this set is 0 because this set E + square root of 5 intersection Z / N is empty for any N because E remember is made up of rational's in 0,1. So if take q rational let us say p / q in 0, 1 and we have p / q + root 5 is some k over N then this implies that root 5 is a rational k over N – p / q this belongs to rational this is a contradiction.

Therefore this this set is empty for any N and so the count is 0 and so in the limit you will get 0 therefore if we try to define our measure E for an obituary subset using the limit formula then this fails the translation invariance property. So if you take the translation if you take the limit formula fails the translation invariance. Therefore, since we want our measures to satisfy finite additivity and translation invariance and non-negativity.

We will try to approximate any obituary set in Rd by elementary sets and see whether by approximation we can achieve some reasonable results.

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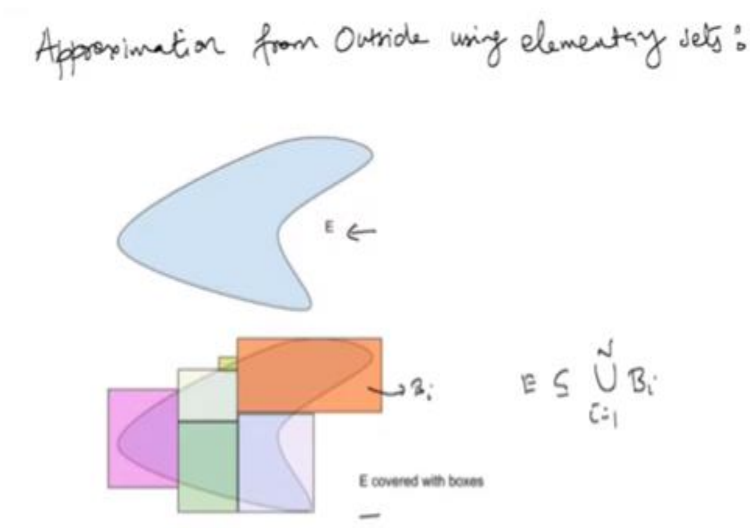

So to illustrate this approximation here we have a picture of an arbitrary subsets E so this is arbitrary subset of R2 and it is definitely not a box or an elementary set. So this is not an elementary set so the idea here is to look at boxes which are lying completely inside E and take the union so you will have. So the union of all these colored boxes is elementary subsets, which is lying entirely inside E.

And if we increase the number of such boxes it is quite possible that E can be approximated from inside with elementary sets this is the union of boxes. So here for example here you can have another box and so on so all these gaps can be filled with more boxes and you can have a nice approximation of E using entirely elementary sets. So now similarly you have can have an approximation of this set E using elementary sets which completely covered E.

So here E is a subset of the union of this colored boxes B r so these are the boxes Bi and if you E can be covered by such boxes then and E can be approximated by from outside by this boxes then we can move to use our already well defined notion of elementary measure both from inside and outside to create a new measure for arbitrary subsets of Rd.

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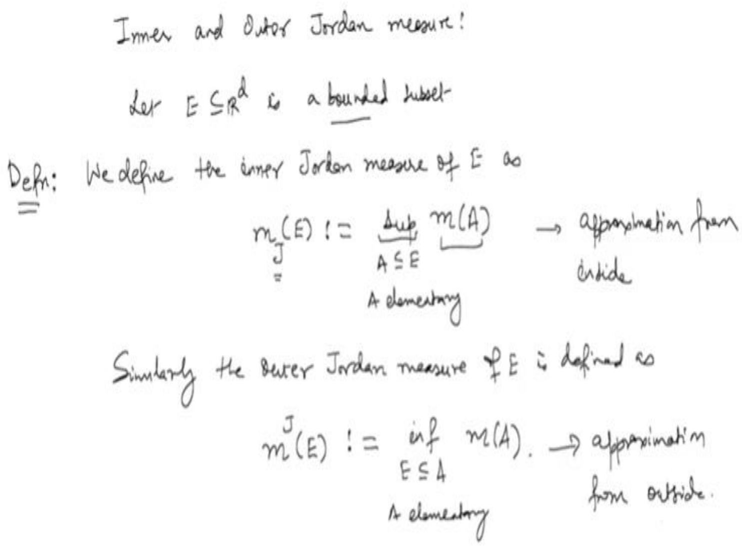

So let us see how this is done so this is the inner and outer Jordan measure. So suppose that E is a bounded subset of Rd which need not be an elementary set is just bounded. So this is the definition we define the inner Jordan measure of E as so we will denote the inner Jordan measure with a lower subset J we will denote it like this is the inner Jordan measure and by definition this is the supremum over all elementary sets A inside E A elementary of the elementary measures of A.

So you are choosing an elementary set A which lies completely inside E you are taking the measure of that set A and you will take a supremum of all such elementary sets and this numbers of m the elementary measures of the sets. Similarly the outer Jordan measure of E is defined as now we will denote it as m the super script J. And this defined as the infimum over E subset of A elementary m A.

So in the first one we will approximate from inside this is approximation from inside and the second one is approximation from outside.

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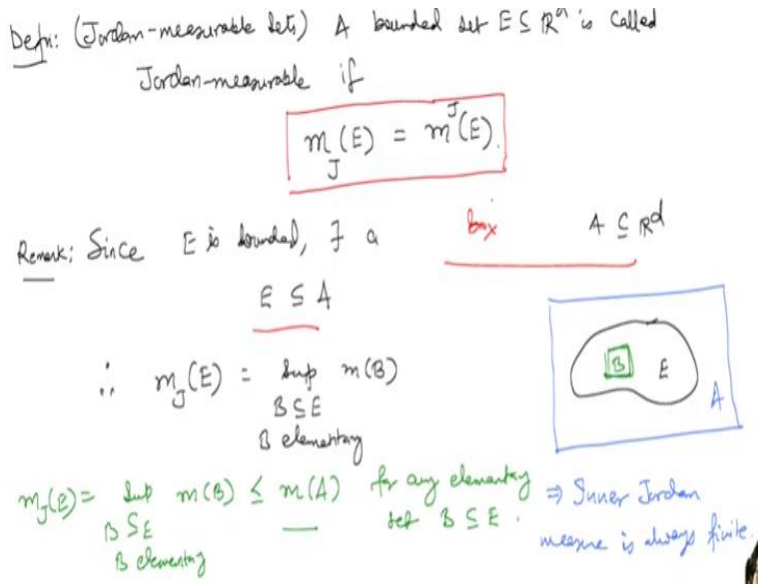

Now we can define what is the Jordan measurable set of Rd Jordan measureable sets so this is the definition? So a bounded set E a subset of Rd is called Jordan measureable if we have the equality of the inner Jordan measure of E with the outer Jordan measure of E. Note that because we have chosen E to be bounded since this is an important remark since E is bounded they are exist an elementary set A of Rd which completely covers E.

So here we have some E then one can always cover AE this E with an elementary set or even a box we can even take a box with they are exist of box A asses that E is completely inside A. Therefore our inner Jordan measure which is the supremium by definition the supremum of the sets the elementary so let us write here B elementary of m B. Now we know that A we have all this B sitting inside E totally therefore by monotonous (()) (15:30) property we have m of B is less than equal to m of A for any elementary set B sitting inside E.

So therefore if you take the supremum over all such the elementary then this is still less than equal to, n of A and this is nothing but with inner Jordan measure of E. Therefore inner Jordan measure is implies that inner Jordan measure is always finite. Similarly one can have the outer Jordan measure will also be always be finite because so similarly outer Jordan measure will also be finite number for any bounded set E.

This is because the outer Jordan measure of E is bi-definition by the infimum over all sets B which contain E and all these B's are all elementary and you are taking the measure of these sets B. But note that since E is bounded there exist a box now let me call it A which contains E as we saw before. There is always a big enough box because A is bounded so big enough box which contains E.

So this box A is now admissible in this collection of elementary sets which contain E therefore this is always less than or equal to m of A which is finite. Therefore we see that both inner and outer Jordan measures are always finite and by definition we will have that a set will be called Jordan measureable if these 2 finite numbers agree which means that the inner Jordan measure is equal to the outer Jordan measure.

So in the next lecture we will see more properties of Jordan measures outer Jordan measures inner Jordan measure and we will give conditions necessary insufficient conditions for the Jordan measurability of a bounded subset of Rd.