Measure Theory
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Module No # 01
Lecture No # 05
Uniqueness of Elementary Measure and Jordan Measurability – Part I

So in the last lecture we saw some basic properties of elementary measures and these are very nice properties and they follow our geometric intuitions very nicely. For example translation invariance finite additivity and so on but we only consider up to now elementary substrates of RF. Now we would like to enlarge our collection of subsets to which a notion of measure can be applied and for this reason we will now consider more subsets of RN bounded substrates which can be approximated by elementary subsets.

And we will see that once we have the right notion of the approximations then the new measures that we can define these are called the inner and outer Jordan measures. They follow they inherit the nice properties of the elementary measures such as translation invariance and finite electivity and finite sub additivity.

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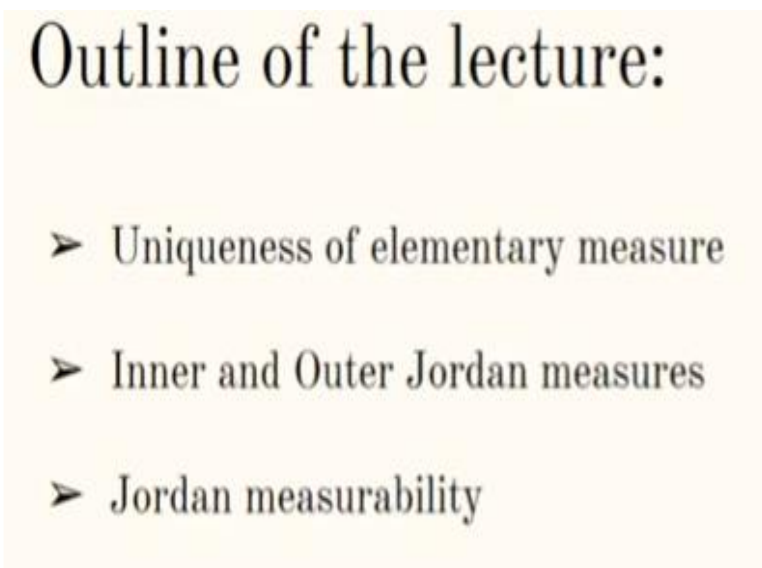

So in this lecture first before going to the inner and outer Jordan measures we will see that the way we have defined the elementary measures or elementary substrates of a RD or RN. If you have any other map which assigns a numerical value a positive or non-negative numerical value to an elementary subset of RN which also satisfies translation invariance non-negativity and finite activity. Then it is related with the elementary measure but up to multiplication by a constant.

So in that sense our elementary measure is the unique measure up to normalization by a constant which satisfies non-negativity translation invariance as well as finite additivity. This fact will be useful later when we study the Jordan measures so once we have seen this uniqueness property or the elementary measure we will then move to define what are the inner and outer Jordan measures? And then we will also define the Jordan measureable subsets of RN.

We will see that our inner and outer Jordan measures are always finite and if these 2 numerical values are equal then we call that set to be Jordan measureable.

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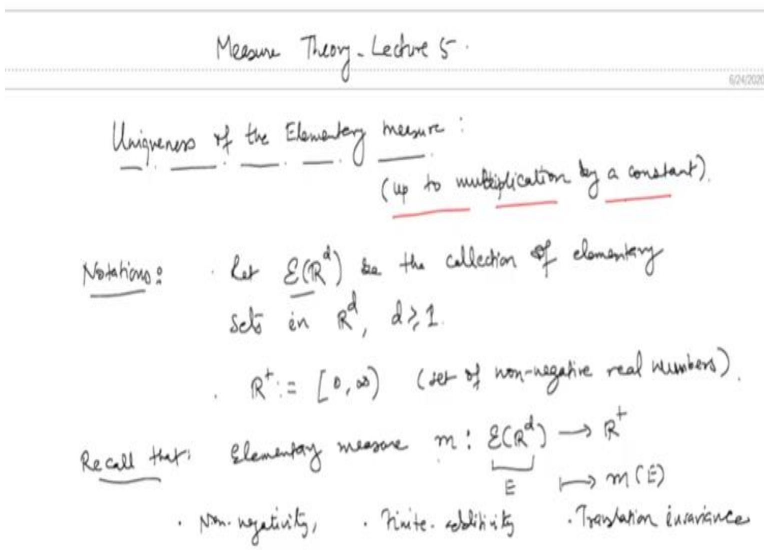

So let us begin with the property of the uniqueness of our elementary measure but this holds only up to multiplication by a constant. So I will explain what is this means and where is the uniqueness coming from? But first let me make a couple of notations that I will use later so first is that let ERd be the collection of elementary sets in Rd, for, d greater than L. So calligraphic d Rd is the collection of elementary subsets in Rd and secondly I will denote R+ to be this semi access from 0 to infinity.

This is a set of non-negative real numbers so now recall that our elementary measure which I now view from this space ERd to R+. So, it takes as input an elementary set E and it gives you the value mE which is a non-negative real number. Now this elementary measure satisfied non-negativity finite additivity and translation invariance. So now if we suppose that if we have another such map which takes as input an elementary set and gives you can output in R+ and it also satisfies non-negativity finite additivity and translation invariance.

Then we will see that that it will relate to the elementary measure by a simple multiplication by a constant. Let us state this theorem.

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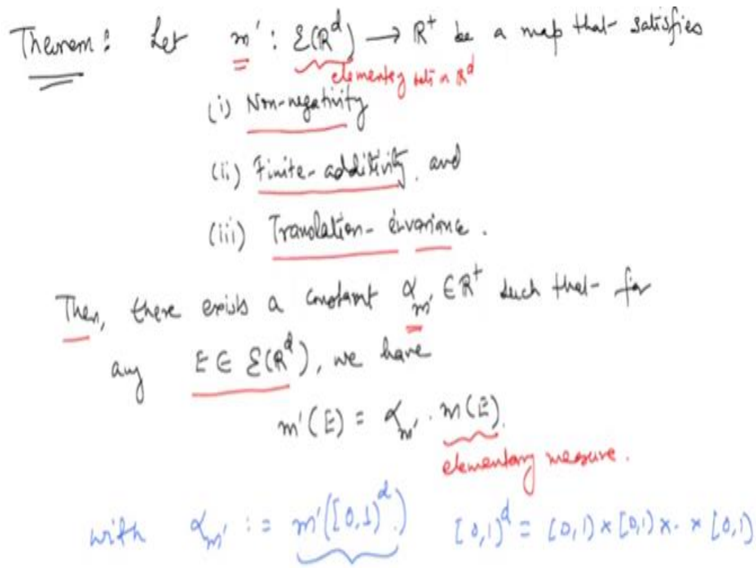

So this is our theorem it says that it m prime is a map from the elementary subsets of Rd to the non-negative real's R+ which satisfies 3 properties. First is non-negativity the second is finite additivity and third is transition invariance. So we have seen that our elementary measures satisfies all this 3 things and now we are now we are assuming that there is another map m prime which also satisfies these 3 properties.

Then this theorem says that they are exists a constant alpha m prime so this constant alpha m prime only depends on m prime. Such that for any elementary set E so this is an elementary set in Rd we have that m prime E = alpha m prime time's m E and this is our elementary measure on the right. And we can even explain further what is this constant alpha m prime so alpha m prime given by m prime of the set 0 1 so this is the hyper queue 0, 1 in d dimensions.

So 0, 1 d is simply the Cartesian product of the intervals 01 in d dimensions so alpha m prime is simply whichever number you have assigned via m prime from this hyper queue 0, 1 to the d.

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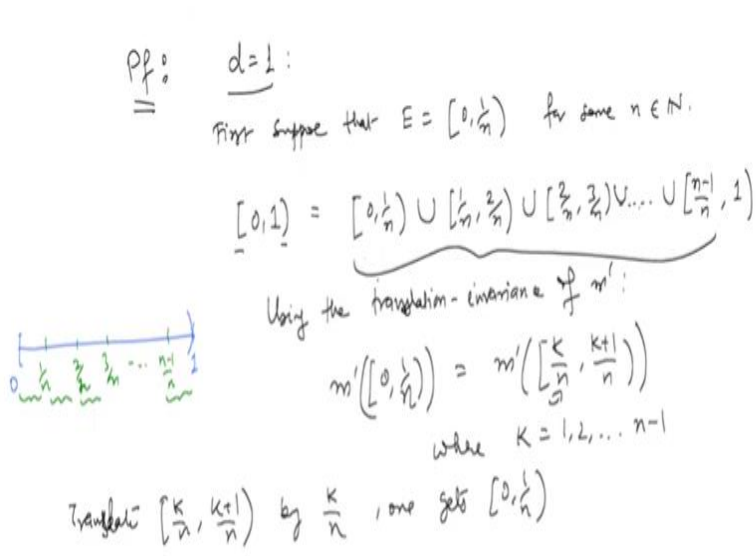

So let us see the proof I will only prove it for d = 1 again so for the real line for the subsets of real line. And I will leave the general proof of arbitrary mentions as an exercise. So let us do this for d = 1 so first suppose that our E is the set 0, 1 / n for sum n sum natural number N. Now let us guide our set 0, 1 closed at 0 open at 1 as the following union I take 0, 1 / n union 1 / n, 2 / n union 2 / n, 3 / n and so on.

And the last one is n – 1 / n, n / n which is 1 so what I am doing here is breaking up the set 0, 1 so we have this set 0,1 which is open at 1 and closed at 0. And I am breaking it up in chunks of size 1 over n so this is 1 / n this is 2 / n 3 / n and so on n – 1 / n. So each one as size n / n and here I will use the translation in variance of m prime. So using the translation invariance of n prime we get that n prime of this set 0, 1 / n = n prime of any set k /n k+1 / n where k ranges between 1, 2 up to n – 1.

Because this is just translating by this number k / n so if you translate this set k / n to k + 1 / n so translate this set by the number k over n 1 gets 0, 1 over n. So by translation invariance we get that the n prime of 0, 1 / n = m prime of each chuck that appears in this decomposition. And now note all these are elementary subsets of R and this is a disjoint union.

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By finite-additivity:

$$m'([0,1)) = m'\left(\bigcup_{k=0}^{n-1}\left[\frac{k}{n},\frac{k+1}{n}\right)\right)$$

$$= \sum_{k=0}^{n-1} m'\left(\underline{\left[\frac{k}{n},\frac{k+1}{n}\right)}\right)$$

$$= \sum_{k=0}^{n-1} m'\left(\left[0,\frac{1}{n}\right)\right) \quad [\text{Translation invariance}]$$

$$= n \cdot m'\left(\left[0,\frac{1}{n}\right)\right)$$

$$\Rightarrow \quad m'\underbrace{\left(\left[0,\frac{1}{n}\right)\right)}_{E} = \underbrace{\frac{1}{n}}_{m([0,\frac{1}{n}))} \cdot \underbrace{m'([0,1))}_{\alpha_{m'}} = \alpha_{m'} \cdot m\underbrace{\left(\left[0,\frac{1}{n}\right)\right)}_{E}$$

Therefore now I am using finite additivity property so by finite additivity we have that m prime of 0, 1 which was m prime of union k / n k + 1 / n k = 0 to n – 1. And this is a disjoint union so this is simply by finite additivity property k = 0 to n – 1 m prime of the set k / n k + 1 / n. And so we have that each set in this sum is the same as m prime time's m prime of 0 1 over m just by translation invariance that we have already seen translation invariance.

And we have the sum ranges from 0 to n -1 so we have n times so therefore this is m time's m prime of 0 1 over n. Therefore we get n prime 0 1 over n = 1 over n times m prime of 0, 1 now this was our alpha m prime for the gas d = 1 this is our alpha m prime. And this is nothing but m of the set of 0 1 over n so this is nothing but alpha m prime time's m of 0, 1 over n. So for this set e we get our required result which is that n prime of e = alpha n prime time's m of E.

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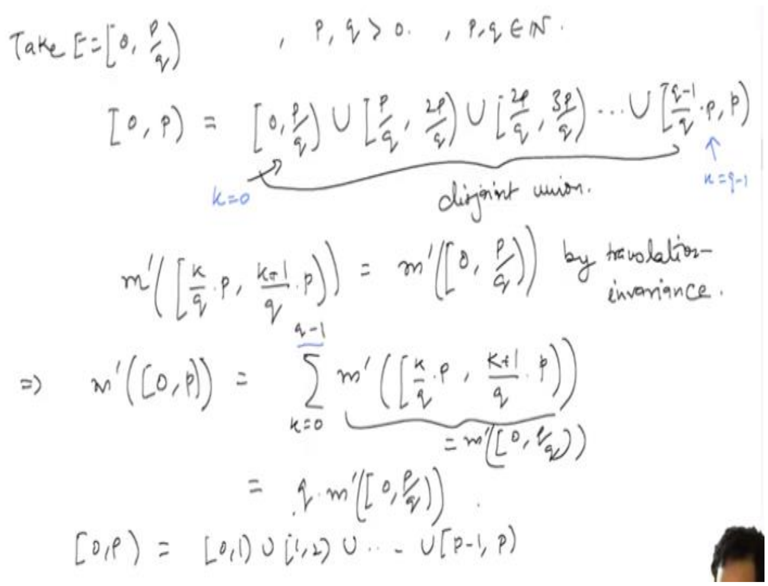

Now I am going to do the same thing but for the set now take 0 p over q so we have done it for 1 over n now we were going to do this for positive rational p over q. So here p and q are greater than 0 so this is our E now and now what I am going to do is take this set 0 to p and now I am going to divide this in chunks of length p over q. So this can be written as disjoint union p over q to p / q union 2p /q 3p/q and so on. And you will end up with a interval q – 1 / q time's p and then q / q time's p q / q time's p which is nothing but p.

So this is now disjoint union and again we will use the same trick to write m prime of k / q time's p k + 1 / q time's p this is the same as m prime of 0 p / q by translation invariance. So therefore if I try to compute the value of m prime for the interval 0 p you will get the sum from k = 0 to q – 1 m prime of k / q times p to k + 1 / q times p. So it begins at this interval when k = 0 you will get a 0 to p / q and at the end then k = q – 1 we will get this internal which is q – 1 over q times p to, p.

Now each of this is equal to m prime 0 to p / q so therefore since you have q as many terms so you have q times m prime of 0 p / q. Now for the left hand side we can divide it in another way which is so here of course p and q are natural numbers. Because I am only considering rational p / q so p and q are natural numbers. So I am also going to write down 0, p as 0, 1 union 1 to union up to union of p – 1 times p -1 up to p.

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By Translation-inv. & finite additivity

$$m'([0,p)) = p \cdot m'([0,1))$$

$$\Rightarrow \quad \underbrace{p\, m'([0,1)) = q \cdot m'([0,p/q))}\; = m'([0,p))$$

$$\Rightarrow \quad m'([0,p/q)) = \underbrace{\frac{p}{q}}_{m([0,p/q))} \cdot \underbrace{m'([0,1))}_{\alpha_{m'}} = \alpha_{m'} \cdot m([0,p/q)),$$

$1/n$, p/q, $r \in \mathbb{R}^+$.

So this is again disjoint union and so if you use translation invariance and finite additivity you will get that m prime of 0, p is now p times m prime of 0, 1. So by these 2 decompositions we will have m prime of 0, 1 times p = q m prime of 0, p / q both and these are both equal to m prime of 0, p. Now I am only going to consider the first equality this implies that m prime of 0, p / q = p / q times m prime of 0, 1.

Now again we have this alpha m prime and this is the elementary measure of the set 0 to p / q. So this is again alpha m prime multiplied by elementary measure of 0 p / q. So first we did it for 1 over n now we have done it for p / q and our next step will be for any real number any positive real number R.

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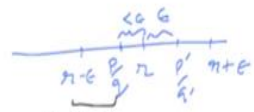
Let $r > 0$, $r \in \mathbb{R}^+$.

Approximation argument: Let $\epsilon > 0$.

$\exists\ p, q > 0,\ p, q \in \mathbb{N}$, such that $r - \frac{p}{q} \leq \epsilon$.

(due to the density of $\mathbb{Q}$ in $\mathbb{R}$)



By Monotonicity (inferred from finite-additivity & non-negativity)

$$\alpha_{m'} \cdot \frac{p}{q} = m'\left(\left[0, \tfrac{p}{q}\right)\right) \leq m'\left([0, r)\right)$$

$$\frac{p}{q} \geq r - \epsilon$$

$$m'\left([0, r)\right) \geq \alpha_{m'} \cdot \frac{p}{q} \geq \alpha_{m'} (r - \epsilon)$$

Similarly $\exists\ p', q' > 0,\ p', q' \in \mathbb{N}$ s.t. $\frac{p'}{q'} - r < \epsilon$

So let us do this for any positive real number r be a positive real number. So for this one, we cannot use this finite additivity directly so we will use an approximation argument. So for this suppose that we have a number epsilon greater than 0 an arbitrary number which you can make as small as possible. Now by the density of rational in the real’s they are exist p, q greater than 0 natural numbers such that r – p / q is less than epsilon.

So we have choosing here so this is our number r and here we are choosing p / q and this distance is less than epsilon. So let us say this is r – epsilon so this distance is less than epsilon so we can do this for any arbitrary epsilon because of the density of rational’s in r. So this is due to the density of rational numbers in R so now by monotonicity remember that our m prime also satisfied non-negativity and finite additivity.

So this is the monotonicity property can be inferred from finite additivity and non-negativity just as did for the elementary measure we can also do it for our new map m prime. So we can prove that monotonicity property of m prime therefore since this intervals 0 to p /q is a subset of the interval 0, r we have that m prime of 0 p / q is less than equal to m prime of 0, 1. Now we already know what our m prime of 0 p / q this is nothing but alpha m prime time’s p / q.

So we have m prime of 0, r is greater than equal or equal to alpha m prime time’s p / q. But note that we have chosen our p / q such that p / q is greater than or minus epsilon from this choice. Therefore this is greater than or equal to so we can even write greater than or equal to so alpha m prime r – epsilon. Similarly they are exist p prime q prime greater than 0 p prime q prime n prime such that you have m prime of 0, r well first of all let me write p prime over q prime – r is less than epsilon.

Now I have chosen approximated this real number r from the right so in this picture you have p prime / q prime this is r + epsilon. So this is again less than epsilon so the distance p prime over q prime and r is less than epsilon.

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By monotonicity: $\quad \frac{p'}{q'} - r \leq \epsilon$

$$\alpha_{m'}(r-\epsilon) \leq m'([0,r)) \leq m'\left(\left[0,\tfrac{p'}{q'}\right)\right) = \alpha_{m'} \cdot \frac{p'}{q'}$$

$$\leq \alpha_{m'}(r+\epsilon)$$

Since $\epsilon > 0$ was arbitrary, we get

$$m'([0,r)) = \alpha_{m'} \cdot r \quad (\alpha_{m'} \cdot \epsilon \text{ is also arbitrary}).$$

$$= \alpha_{m'}\, m([0,r))$$

$\Rightarrow$ For any interval $I = [a,b)$ by translation-invariance

$$m'(I) = m'([0,b-a)) = \alpha_{m'} \cdot (b-a) = \alpha_{m'}\, m(I)$$

And again I can do a similar analysis so by monotonicity we get that m prime of 0, r is less than or equal to m prime of 0 p prime / q prime. And this is nothing but m prime time’s p prime / q prime but since we have chosen p prime / q prime – r is less than or equal to epsilon. So this is less than or equal to alpha m prime time’s r + epsilon and we have already found that this is less than or equal to alpha m prime time’s r – epsilon.

So we have seen this kind of argument before where we approximate from both sides so since epsilon greater than 0 was arbitrary we get that m prime 0, r equal to alpha m prime time’s r. Because this number alpha m prime time’s epsilon is also arbitrary so now we have proved our of course this r can be written as m of 0, r. So now we have proved it for any interval of the form 0, r now this easily implies for any interval I of the form a, b then you can shift by translation invariance you can shift starting point to 0.

So by translation invariance m prime of I = m prime of 0, b – a I am just shifting by a so if you subtract a on both ends will get 0 and b – a. And this I nothing but alpha m prime time's b – a because b – a is real number and this is alpha m prime time's m of I. So therefore we can do this for interval and now the only thing left is to show this for any union of intervals.

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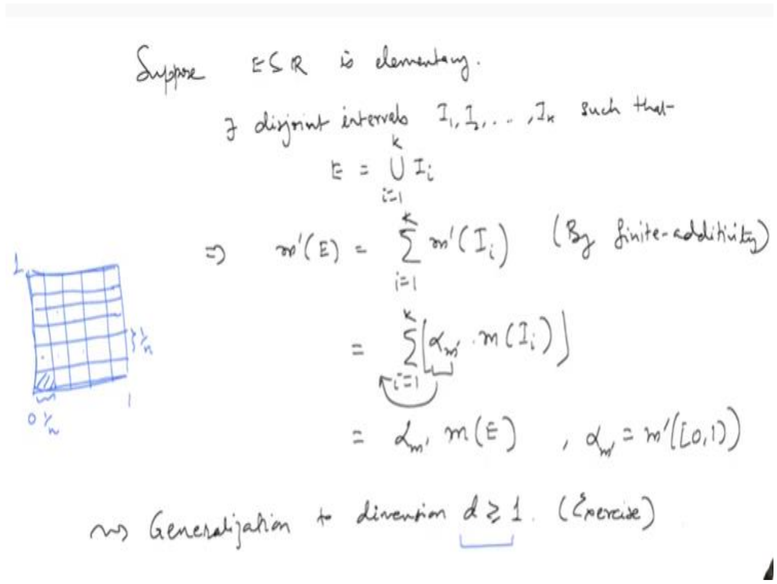

So suppose that E is elementary so then remember that in the previous lecture we will prove that they are exist disjointing intervals I1, I2, Ik such that E is the union of these I is from i = 1 to k. So now this is a disjoint union and so therefore m prime of E = the sum i = 1 to k m prime of Ii by finite additivity property of m prime. But now we know that this is equal to alpha m prime time's m of Ii and this is alpha m prime and by the finite additivity property of our elementary measure this is m of E.

You can take this constant outside the summation so we have shown that m prime of E = alpha m prime of mE where alpha m prime is simply the value m prime of 0, 1. And this kind of argument can be generalized to arbitrary dimensions but there you have to be little bit careful because you have to chop your mE box into many chunks each having side 1 over n. So dimension d greater than equal to 1 this is left as an exercise.

So for example in dimension 2 if you have a box of size so let us say this is 0, 1 + 0, 1 then you have to shop this up into size 1 over n in y direction and also in the x direction 1 over m. So now you have each box has the same value m prime for any of this boxes that appearing this grid form. So this is not very difficult and I will leave it to you as an exercise for any dimension d okay. So now we have seen that our elementary measure is unique or to multiplication by a scalar C.