Measure Theory
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Module No # 14
Lecture No # 75
Differentiation theorem for general monotone functions and Second fundamental theorem

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$$f_Q(b^j_{k,n}) \geq f_Q(a^j_{k,n})$$

$$\Leftrightarrow \quad f(b^j_{k,n}) - Q b^j_{k,n} \geq f(a^j_{k,n}) - Q a^j_{k,n}$$

$$\Leftrightarrow \quad f(b^j_{k,n}) - f(a^j_{k,n}) \geq Q \underbrace{m(I^j_{k,n})}_{(b^j_{k,n} - a^j_{k,n})}.$$

<u>Claim:</u> i) If $U_n = \bigcup_{k,j \geq 1} I^j_{k,n}$, then,

$$(E_{q,Q} \cap I_n) \subseteq U_n \subseteq I_n.$$

ii) $m(U_n) \leq \frac{q}{Q} m(I_n)$.

So now we have this 2 inequalities which will help us estimate our Dini derivatives. So I claim that if I said u n to be the union of k and j greater than equal to 1 I k n j so I am even taking the union over all k and all j then E q Q intersection with the set I n. Remember that I n was one of the open intervals considering the set u open set which covered E q Q and on which applied the first application of the rising sun lemma. And this set is a subset of u n and u n is the subset of I n the measure of u n is less than or equal to q over Q times the measure of I n. So let us see why these claims are true.

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(1) Let $x \in I_n \setminus U_n$, and let $y > x$; $y \in I_n$.

then, $f_Q(y) \le f_Q(x)$.

$$\Rightarrow f(y) - Qy \le f(x) - Qx$$

$$\Rightarrow \frac{f(y) - f(x)}{y - x} \le Q.$$

$$\Rightarrow \underbrace{\limsup_{\substack{h \to 0 \\ h > 0}} \frac{f(x+h) - f(x)}{h}}_{\overline{D^+}f(x)} \le Q. \Rightarrow x \notin E_{q,Q} \cap I_n.$$

$$\Rightarrow E_{q,Q} \cap I_n \subseteq U_n.$$

Now to prove the first part let x be a point taken outside u n but inside I n and let y be a point strictly greater than x inside y n inside I n. Then what we get is that f of q of y is less or equal to f Q of x again by the rising sun lemma because these are points in the sun. So this implies that f y – Q y is less than or equal to f x – Q x and this is nothing f y – f x over y – x is less than or equal to Q. So this implies that Lim sup of h going to 0 with h positive f x + h – f x over h is less than or equal to Q and this is nothing the upper Dini derivative of the positive part.

This is less than or equal to q which implies that x does not belong to e q Q intersection I n because on this set the Dini derivative must be strictly greater than Q. So this means that E q Q intersection is the subset of u n. So this proves the first part and for the second part we argue as follows.

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(ii) $m(\mathcal{U}_n) = \sum_{k,j\geq 1} m(I^j_{k,n})$

$$\leq \frac{1}{Q} \sum_{k,j\geq 1} \left(f(b^j_{k,n}) - f(a^j_{k,n})\right)$$

and since f is monotonically non-decreasing,

$$\leq \frac{1}{Q} \sum_{k\geq 1} \left(f(b_{k,n}) - f(a_{k,n})\right).$$

$$\leq \frac{q}{Q} \sum_{k\geq 1} m(I_{k,n}) = \frac{q}{Q} m(E_n) \leq \frac{q}{Q} m(I_n).$$

$$E_n = \bigcup_{k\geq 1} I_{k,n} \quad (\text{disjoint})$$

So for the second part we have m of u n which is nothing but the sum k j greater than equal to 1 m of I k n j and m of I k n j is less than or equal to 1 over Q. Again sum over k greater than equal to 1 f of b k n j – f of a k n j we have already established this relation and since f is monotonically non-decreasing we have that this is less than or equal to 1 over Q we can sum over j and if you just sum over j. Then, is going to be less than or equal to f of b k n – f of a k n and now again because of the rising sun inequality that we had earlier.

This is less than or equal to q over Q k greater than or equal to 1 m of I k n but this is the set q over Q m of E n and this is less than or equal to q over Q m of I n, This is a disjoint because E n was the disjoint union of I k n k greater than equal to 1 disjoint union. So we get our result that m u n is less than or equal to q over Q over times m of I n. And now we can finish the proof.

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\begin{aligned}
\underline{m(E_{q,Q})} &= \sum_{n \geq 1} m(\underset{\underset{U_n}{\cap}}{E_{q,Q} \cap I_n}) \\
&\leq \sum_{n \geq 1} m(U_n) \leq \frac{q}{Q} \sum_{n \geq 1} m(I_n). \\
&= \frac{q}{Q} m(U) \underset{=}{<} \left(\cancel{\frac{q}{Q}}\right) m(E_{q,Q}) \cancel{\left(\frac{Q}{q}\right)} = \underline{m(E_{q,Q})}.
\end{aligned}
$$

which is a contradiction.

This finishes the proof for continuous, MND fns.

So we have m of E q Q this is equal to the sum of m E q Q intersection I n greater than equal to 1 and this is less than or equal to. So this is the subset of u n remember and so this is the sum over m of u n which is less than or equal to q over Q n greater than equal to 1 m of I n. And now this I n’s are precisely the decomposition of the open set u that we took so this is nothing but q over Q of m of u. But m u was chosen so that this is less than q over Q times m E times Q over q.

So this cancels out and so you just get m E so m E q Q here also m e E q Q and so they have a strict inequality between m E q Q and m E q Q which is a contradiction which proves the result. So this finishes the proof for continuous so differentiation theorem for continuous monotonically non-decreasing functions it is establishes almost everywhere differentiability for such functions. And now I am just going to state what happens for monotonically non-decreasing functions which might have discontinuities.

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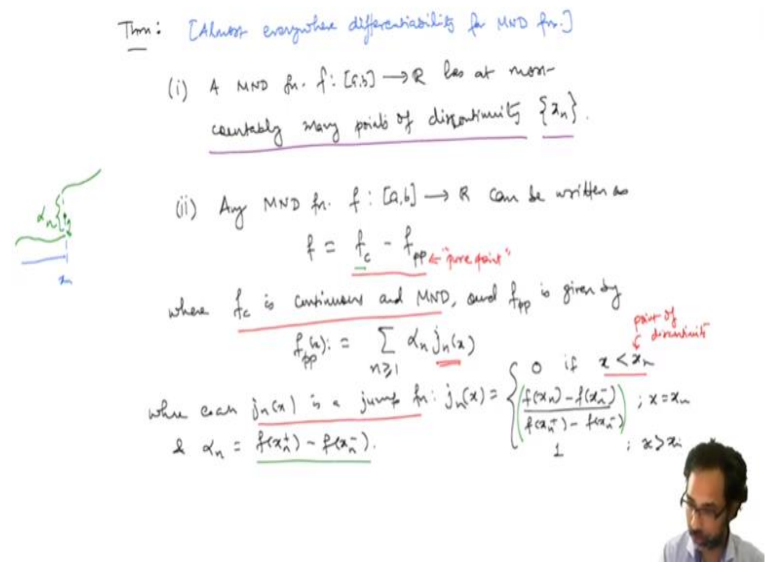

Now finally let me just take the theorem where we can drop the condition for continuity of the monotonically non-decreasing function. And this is a sort of the structure theorem for multiple monotonically non-decreasing functions. So, the first one says that if you have a monotonically non-decreasing function f then it as countably many points of discontinuity at x n. So it is a collection of countably many points in a, b where function f might be possibly discontinuous.

And secondly it can be expressed as the difference of 2 functions f c and f p p where f c is continuous and monotonically non-decreasing and f p p. So p p stands for pure point and it is a sum of countably many what are called jump functions j n x. So each jump function is given by this formula j n x is 0 if x is less than x n. So this is the point of discontinuity and at the point of discontinuity it takes the fractional value of the jump that f has.

So for example so if this is x n then on f comes from the left of x n but it might take a value higher than strictly higher than the limit the left hand side limit. And then again because it is increasing there will be a right hand side limit. And so this different this fraction of the entire length is given by this formula here. And so the jump functions are encoding this kind of these finite discontinuities in the function.

And so this f p p is a sum of countably many such jump functions with amplitude precisely this alpha n which, is the difference of left hand right hand side limits. So in fact this pure point function f p p is differentiable almost everywhere and vanishes the derivative vanishes almost everywhere. And since we have already prove that this one is since it is continuous and monotonically non-decreasing it is differentiable almost everywhere. So then f is differentiable almost everywhere so for all this proof's I will refer you.

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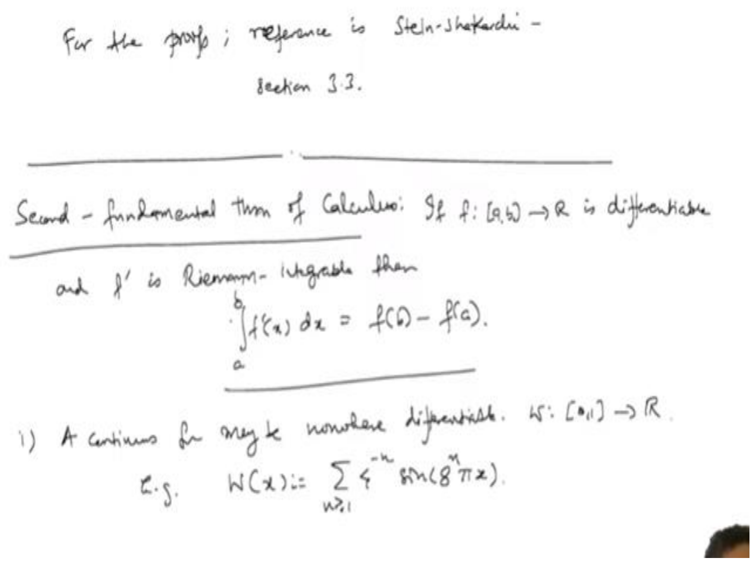

So for the proof's reference is Stein and Shakarchi's book Shakarchi section 3.3 where you will find the detail proof for these results. So this finishes our discussion for the differentiability theorems for functions which are of bounded variation as well as functions which are monotonically non-decreasing. Finally let me just say some concluding remarks about the second fundamental theorem of calculus.

So it says that if f from a, b to R is differentiable and f prime is Riemann integrable then the integral a to be of, f prime x d x is nothing but f b – f of a. So we can try to ask whether even if we have let us say a continuous function which is in l1 over a, b and which is differentiable almost everywhere in a, b then whether such a formula holds. First of all a continuous function may be nowhere differentiable an example is given by the so called Weiestrass function which is given by this sum sin of a to the power n Pi x.

And so this is defined over 0, 1 to R and one can show that this is nowhere differentiable in this interval 0, 1. On the other hand there are continuous functions which might be differentiable almost everywhere but for which these kinds of functions these kinds of equations do not hold if we replace the Riemann integral by the Lebesgue integral.

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(ii) Even if $f : [a,b] \to \mathbb{R}$ is diff. a.e. and $f' \in L^1([a,b])$, then in general

$$\int_{[a,b]} f' \, dm \neq f(b) - f(a)$$

e.g. the Cantor-function. ($f' = 0$ a.e. $f(0) = 0$, $f(1) = 1$).

So even if we have a differentiable almost everywhere differentiable function f for which the derivative is in l1. Then in that case it is not true that the Lebesgue integral of a prime give you f b – f a. So a counter example given by the contra function on the defined using the contra set and for this function the derivative is 0 almost everywhere because it is constant almost everywhere. So the derivative is 0 almost everywhere but f 0 is 0 and f1 is 1. So the left hand side is 0 but the right hand side is 1.

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Defn. [Absolutely Continuous fn] : A fn. $f : [a,b] \to \mathbb{R}$ is said to be absolutely continuous if given $\epsilon > 0$, $\exists \, \delta > 0$ s.t. if $\{(a_k, b_k)\}_{k=1}^{n}$ is a finite collection of disjoint open intervals in $[a,b]$ s.t. $\sum_{k=1}^{n} (b_k - a_k) < \delta$, then we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

Rk: Absolutely continuous fns. are continuous.

We make the following definition for absolutely continuous functions. So this is a function f and it is said to be absolutely continuous if given epsilon greater than 0. There exists delta greater than 0 such that we have the following condition that if there exists finitely many disjoint open intervals a k, b k. So this is the finite collection of disjoint open intervals in a, b such that the total length of this union is bounded by delta then the variation.

So this is the sort of a variation of f over this intervals is less the epsilon. So these are called absolutely continuous functions. Now note that absolutely continuous functions are continuous and in fact uniformly continuous on a, b. And the point is that the absolutely continuous functions the analog of second fundamental theorem of calculus holds for the Lebesgue integral.

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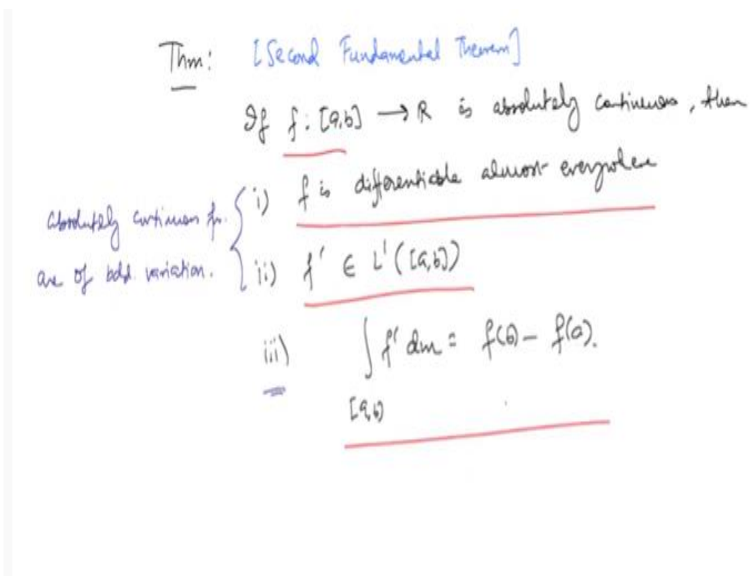

And so the point is here is that the second fundamental theorem of calculus will hold for absolutely continuous functions. So if you have an absolutely continuous function on a compact interval a, b. Fist that f is differentiable almost everywhere f prime is an absolutely integrable function on a, b and thirdly that the integral over a, b of, f prime is precisely f b – f a. Now the first 2 are actually consequences of the fact that absolutely continuous functions are of bounded variation.

So we have shown that for function of bounded variation it is differentiable almost everywhere. It is also true that the derivative of the bounded variation function which is defined almost everywhere is in fact an l1 function. So this third part is the absolute continuity place a role that the second fundamental theorem of calculus holds. So we come to the end of this topic about differentiation theorems.