Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Module No # 14
Lecture No # 74
Differentiation theorem for monone continuous functions

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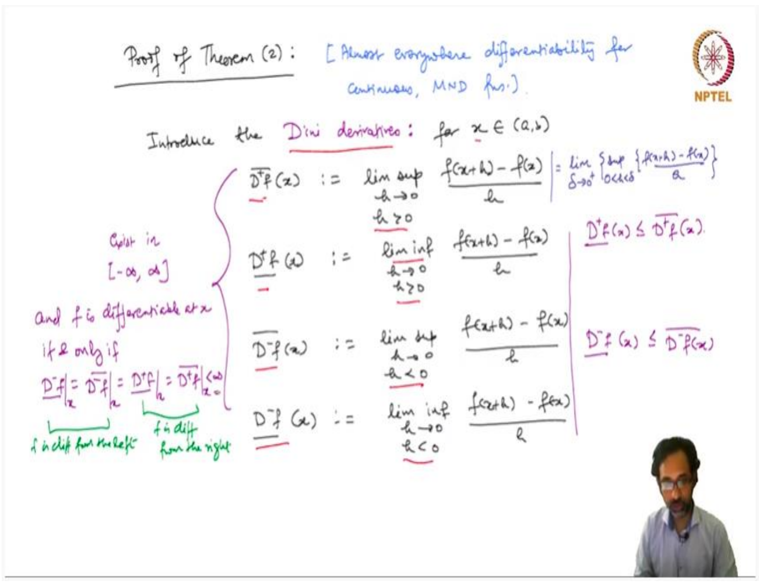

So let us being the proof of theorem 2 about almost everywhere differentiability for continuous monotonically non-decreasing functions and for this we introduce what is called the Dini derivatives. And if you take a point x in a, b interior point in a, b then the first one d + f upper bar this is equal to the Lim sup as x tends to 0 of this expression f x + h – f x over h that appears in the definition of the derivative but rather than taking limit as s tends to 0 we take the Lim sup as s tends to 0 where h is strictly positively.

So h tends to 0 from above and you take the limb soup so by definition this Lim sup is equal to the limit of the supremum of this expressions f x + h – f x over h and the supremum is taken over all h in an interval 0 to delta. And then you let delta go to 0 from the right so this is by definition this Dini derivative d + f upper bar and similarly one defines Lim inf so the second one is d + f lower bar this is Lim inf of h tends to 0 with positive values.

Similarly d – f upper bar is Lim sup as s tends to 0 but now h is less than 0 and d – f lower bar which is Lim inf of s tends to 0 with the h taking negative values. So all these Dini derivatives so, they exist in the extended real numbers minus infinity to plus infinity and f is differentiable at x. If and only if all the Dini derivatives coincide so if and only if d – f upper bar. So let us start with the lower d – f upper bar so all of these are at x then d + f lower bar at x and d + f upper bar at x.

So all of these when they coincide then f is differentiable because the limits will exists if only the plus ones coincide then f is differentiable from the right. And if and only if then minus ones coincide then d is differentiable from the left. So if only these 2 coincide then f is differentiable from the left and if these 2 are equal then f is differentiable from the right and so on and we also have some inequalities

So first one is that so these are quite trivial d + f lower bar is less than or equal to d + f upper bar of x. Because one is the Lim inf the other one is the Lim sup and similarly we have that d – f lower bar of x is less than or equal to d – f upper bar of x. So these are the Dini derivatives and we will show that for almost every x in this interval a b so we can leave out the points a, and b because they are of measures 0.

But we will show that for almost every x in this open interval a b all these Dini derivatives coincide. And so the function when it is continuous and monotonically non-decreasing it will be differentiable for almost every x in a b. I should also add here that it is not sufficient to have equality for all these Dini derivatives for f to be differentiable. Since they exist in the extended real numbers we should also impose that all of these are finite as well as that they coincide.

So this finiteness is also important so to show the proof of almost everywhere differentiability we will make some claims.

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Claims: i) All Dini Derivatives are measurable fns.. [Ex]

ii) $\overline{D^+}f(x) < \infty$ for x - a.e. in (a,b)

iii) $\overline{D^+}f(x) \leq \underline{D^-}f(x)$ for x - a.e. in (a,b)

$\overline{D^+}$: $\limsup_{h \to 0, h>0}$ $\quad$ $\underline{D^-}$: $\liminf_{h \to 0, h<0}$

(iii) $\Rightarrow$ (iii)$'$: $\overline{D^-}f(x) \leq \underline{D^+}f(x)$ for x-a.e. in (a,b),

(obtained by applying (iii) to the fn. $\tilde{f} : [-b,-a] \to \mathbb{R}$,

$\tilde{f}(x) = -f(-x)$) [Check].

(ii) & (iii) $\Rightarrow$

$$\overline{D^+}f(x) \overset{(iii)}{\leq} \underline{D^-}f(x) \leq \overline{D^-}f(x) \overset{(iii)'}{\leq} \underline{D^+}f(x) \leq \overline{D^+}f(x) \overset{(ii)}{<} \infty$$

for x-a.e. in (a,b)

So these are 3 claims that we are going to make. So, the first one is that all these Dini derivatives are measureable functions. So this I am going to leave as an exercise to show that these are in fact measureable functions using similar arguments that we used to show the Lim sup of a sequence of functions and the Lim inf of a sequence of functions are measureable except that here has to choose when h goes to 0 you have to choose a countable sequence.

So this is not so difficult and I leave it to you as an exercise the second one claims that the upper Dini derivative the limb soup for the positive part is finite almost everywhere in a b. And the third one claims that the d + upper bar meaning that Lim sup taken from the values of h strictly greater than 0 is less than or equal to the Dini derivative d – f lower bar which is the Lim inf taken so this is the Lim sup h tends to 0 from other positive side.

And this is the Lim inf h tends to 0 from the negative side so the d plus upper bar is less than or equal to d – f lower bar and in fact this point 3 implies that d minus upper bar f x is also less than or equal to d + f lower bar x for x almost everywhere in a, b. So this can be obtained by applying this third result to the function f, tilde defined on the interval - b – a, to r given by f tilde x equals – f of – x.

So when you take –x you land up in the interval a b and so you can apply f but then you can apply f then you also again apply an minus sign. And Lim inf from the Lim sups getting to change and so on so check this that 3 implies that the Lim sup for the negative Dini derivative is less than equal to Lim inf for the positive side of the Dini derivative for almost every X in a b. And now that we have these 2 so we have this chain of inequalities so this means so 2 and 3 imply together that first d + f upper bar at x less than or equal to d – f lower bar at x.

So this is by 3 and then we have is less than or equal to d – f upper bar at x just because this is Lim inf on the left and the Lim sup on the right. And then again by the consequence of this so let me call this 3 prime so this is by 3 prime we have that this less than or equal to the d + f lower bar. And this is again less than or equal to the d + f upper bar of x and this is finite by the second part.

So we see that we have a chain of inequalities where the left one and right one coincide and so all of these are equal rest and for this holds for x almost everywhere in a, b and this will give as the result. So we will show firs that the second part d + f upper bar is finite and then the third part for this inequality between d plus upper bar and d minus lower bar.

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(ii) To show: $\overline{D^+}f(x) < \infty$ for x-a.e. in (a,b)

Consider for a fixed $\lambda > 0$:

$$E_\lambda := \{x \in (a,b) : \overline{D^+}f(x) > \lambda\}.$$

We will show: $m(E_\lambda) \le \dfrac{f(b)-f(a)}{\lambda}$

we let $\lambda \to \infty \Rightarrow m\Big(\underbrace{\bigcap_{\lambda>0} E_\lambda}\Big) = 0$.

$$E_\infty := \{x \in (a,b) : \overline{D^+}f(x) = +\infty\} = \bigcap_{\lambda>0} E_\lambda.$$

$\Rightarrow m(E_\infty) = 0$.

So let us start with the proof of 2 so we to show that the upper Dini the positive Dini derivative for the upper bar is finite for x almost everywhere in a b. And to show this consider for a fixed lambda positive the set e lambda defined as the set of x all x in a, b such that d + f upper bar x is greater than lambda. And we will show that the measure of E lambda the Lebesgue measure of e lambda is less than or equal to f b – f a over lambda.

And so if we take if we let lambda go to plus infinity this implies that m, E lambda well the intersection of all these m, E lambda positive this is equal to 0 because the right hand side goes to 0 as lambda goes to plus infinity. But note that this is precisely the set infinity which is the set of points x in a, b such that d + f upper is equal to infinity positive infinity. And this is equal to the intersection of all these E lambda's.

So this will show that m, E infinity is equal to 0 so we would have proven that the upper Dini derivative is finite for almost every x in a, b. So we will need to show this inequality and to show this we will use the rising sun lemma.

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Apply the rising sun-lemma to the fn. $g : [a,b] \to \mathbb{R}$.

$$g(x) := f(x) - \lambda x.$$

$\Rightarrow$ $\exists$ a countable collection of disjoint relatively open intervals $\{I_k\}_{k\geq 1}$

s.t. if $I_k = (a_k, b_k)$, then $g(b_k) = g(a_k)$

and if $I_k = [a, b_k)$ then $g(b_k) \geq g(a)$.

$\left.\right\} g(b_k) \geq g(a_k)$

$$g(b_k) \geq g(a_k) \iff f(b_k) - \lambda(b_k) \geq f(a_k) - \lambda a_k$$

$$\iff f(b_k) - f(a_k) \geq \lambda(b_k - a_k).$$

$$\iff m(I_k) \leq \frac{1}{\lambda}\left(f(b_k) - f(a_k)\right).$$

So apply the rising sun lemma to the function g x defined as f x – lambda times x. So g is again a function from a, b to r and defined with the following formula f x – lambda x. So the rising sun lemma implies that there exist a countable collection of disjoint relatively open intervals I, k such that if I k is equal to a k, b k then g of b k equals g of a k. And if I k equals a b k then g b k is greater than or equal to g a k or rather g a.

So in both cases we have g b k is greater than or equal to g a k right and if we write this inequality g b k greater than or equal to g a k. This is the same as saying that f b k – lambda b k is greater or equal to f a k – lambda a k which is the same as saying that f b k – f a k is greater than or equal to lambda times b k – a k. In other words the measure of the interval I k is less than or equal to 1 over lambda f b k – f a k and now the set E 1 over lambda f b k – f a k.

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On the other hand,

$$\{x \in (a,b) : \overline{D^+}f(x) > \underbrace{\lambda}_{>0}\} =: E_\lambda \subseteq \bigcup_{k \geq 1} I_k =: E$$

$\Rightarrow \exists\, h > 0$ s.t. $\dfrac{f(x+h) - f(x)}{h} > \lambda > 0$

$\Rightarrow f(x+h) - f(x) > 0.$

$\Rightarrow x \in E$ (used in the rising sun lemma).

$$\Rightarrow m(E_\lambda) \leq \sum_{k \geq 1} m(I_k) \leq \sum_{k \geq 1} \frac{1}{\lambda}(f(b_k) - f(a_k)) \leq \frac{1}{\lambda}(f(b) - f(a))$$

[Ex.]

$$\Rightarrow m(E_\infty) = 0.$$

$$\Rightarrow \overline{D^+}f(x) < \infty \text{ for } x\text{-a.e. in } (a,b).$$

Now on the other hand the set E lambda is the sub set of union of all I k's k greater than equal to 1 because remember that this was the set of points x in a b such that d + f bar x is greater than lambda and since lambda is strictly positive this means that there exist an h positive such that f x + h – f x over h is greater than lambda which is positive which means that f x + h – f x is positive and so this x belongs to the set E used in the rising sun lemma.

And this is on the right is it precisely the set E so each of these E lambda's is a sub set of E this means that the measure of E lambda is less than or equal to the sum of this I k's k greater than or equal to 1. This is simply by countable additivity countable sub additivity rather and then we have that this is less than or equal to k greater than or equal to 1 over lambda f b k – f a k by the inequality that we just proved here.

And this is less than or equal to 1 over lambda f b – f a so by taking a sort of a partition of the interval a b you can fill in other points and make it this is a telescoping series so that this terms f a k and f s k + 1 they will or rather f b k and f a k + 1 they will cancel and we are going to be left with f b – f a. So I am going to leave this as an exercise so this inequality where you move from the sum of f b k – f a k to f b – f a I will leave this as an exercise for you to check.

And this shows what you wanted to prove and this implies that measure of E infinity equals 0 so this shows that the upper Dini derivatives the positive 1 is infinite for x almost everywhere in a, b.

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(iii) To show: $\overline{D^+}f(x) < \underline{D^-}f(x)$ for $x \in (a,b)$ a.e

Let $Q > q$ be rational numbers, and consider the set

$$E_{q,Q} := \{x \in (a,b) : \overline{D^+}f(x) > Q > q > \underline{D^-}f(x)\}$$

We will show that $m(E_{q,Q}) = 0$

$$\Rightarrow m\Big(\bigcup_{\substack{Q>q \\ q,Q \text{ rationals}}} E_{q,Q}\Big) = 0$$

$$\| $$

$$\Rightarrow m(\{x \in (a,b) : \overline{D^+}f(x) \geq \underline{D^-}f(x)\}) = 0.$$

$$\Rightarrow \overline{D^+}f(x) < \underline{D^-}f(x) \text{ for } x \in (a,b) \text{ a.e.}$$

So now we come to the third part which is to show that the upper Dini derivative for the positive side is less than the lower Dini derivative for the negative side for x in a, b for almost everywhere. So to do this we consider 2 rational numbers Q and q so these are rational numbers and we consider the set E q Q which is the set of all x in a b for which the upper Dini derivative for the positive side is greater than Q this Q this less than q. And this is greater than the lower Dini derivative on the negative side.

So we will show that the measure of E q Q is equal to 0 which will imply that union of the sets E q Q the union over all rationals Q greater than q rationals. This is also going to be 0 and this implies that the measure of the set x in a b such that b + f x is greater than or equal to d – f lower bar this set as also measure 0. Because this is precisely this set this union of the sets e q Q where you take union over all rationals. And so this will show that d + f upper bar is less than d – f lower bar for x in a, b almost everywhere.

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Suppose that $m(E_{q,Q}) > 0$ (and we shall arrive at a contradiction)

By outer-regularity, and since $\frac{Q}{q} > 1$, $\exists$ an open set $\underline{U \supseteq E_{q,Q}}$, $U \subseteq (a,b)$;

such that $m(U) < \underbrace{m(E_{q,Q})}_{>0} \cdot \left(\frac{Q}{q}\right)_{(>1)}$.

Now let $U = \bigcup_{n \geq 1} I_n$ countable disjoint union of open intervals.

Apply the rising-sun lemma to the continuous fn.

$$\tilde{f}_q : -I_n \to \mathbb{R}$$

$$\tilde{f}_q(x) := f(-x) + qx.$$

So to do these suppose that the measure is strictly positive and we shall arrive at a contradiction. Now we start by taking by outer regularity and since this number Q over q is strictly greater than 1 Q versus q strictly greater than 1. There exist an open set u containing E and u is the subset of this open interval a, b such that the measure of u is strictly less than measure of m E q Q times Q over q. So this is strictly greater than 1 and so by this infimum property one can find such an open set for which this holds because we have assumed this to be strictly positive.

Now let u be written as a countable union of intervals I n so countable union disjoint union of intervals and we apply the rising sun lemma to the continuous function f tilde q which is defined on the set -I n to R. And this function tilde q on-n is defined by f of – x so when you take – x lambda in I n and I n is a subset of a b. so you can happy f + q times x.

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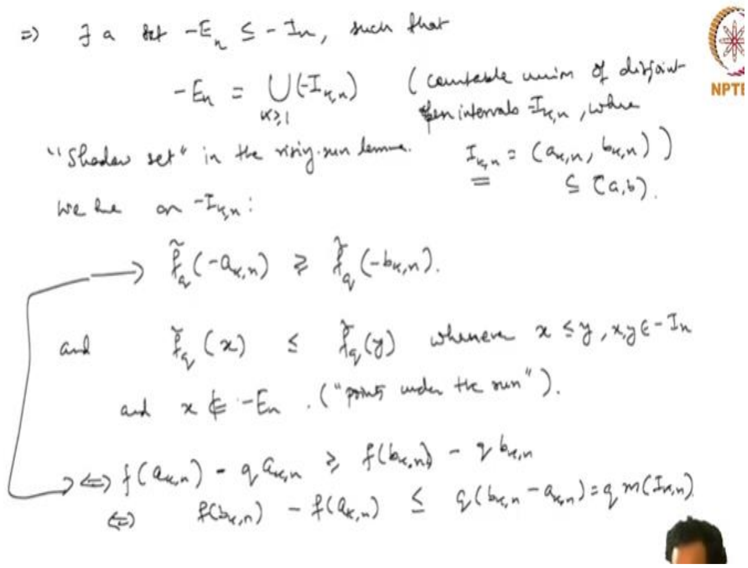

So when you apply the rising sun lemma this implies that there exist a set E – E n inside – I n such that – E n can be written. So first of all it is an open set and it can be written as countable union of disjoint open intervals – I k n where I k n is the interval a k n to b k n is a subset of a b. And so that we also have so this is the shadow set in the rising sun lemma where the sun rises do not hit and whenever you have this I k n's we have on I k – I K n f tilde q of – a k n is greater than or equal to f tilde q of – b k n.

And f tilde q of x is less than or equal to f tilde q of y whenever x is less than or equal to y and they both belong to – I n. So x y in – I n and x y so, x and x does not belongs to I k or rather - E n. So only on the end points of the intervals – I k n we have greater than or equal to sin and for the rest of the values outside – E n we have less than or equal to sin. These are the sets under points under the sun in the rising sun lemma.

And if we unpack this inequality here we get f tilde so f tilde q is f of a k n + q times a k n which is greater than or equal to f of b k n + q times B k n. And so this is the same as saying that f of b k n – f of so it should be minus here and also here because we are taking f tilde q – a k n. So this is the same as saying that f b k n – f a k n which is less than or equal to q times b k n – a k n. And this last term is nothing but the measure of I k n so this is the first inequality that we get and now we will again apply the rising sun this time on the set I k n.

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Apply the rising-sun lemma on $I_{k,n}$ for the fn.

$$f_Q : I_{k,n} \to \mathbb{R}$$

$$f_Q(x) := f(x) - Qx.$$

$\Rightarrow$ $\exists$ a set $E_{k,n} \subseteq I_{k,n}$; $E_{k,n} = \bigcup_{j \geq 1} I_{k,n}^j$.

($I_{k,n}^j$ are disjoint open intervals)

and if $I_{k,n}^j = (a_{k,n}^j, b_{k,n}^j)$

$\Rightarrow$ $f_Q(b_{k,n}^j) \geq f_Q(a_{k,n}^j)$ —

and for any $x \notin E_{k,n}$ (shadow set), for $y \geq x$,

$$f_Q(y) \leq f_Q(x).$$

Now apply the rising sun lemma on I k n for the function f q we find on I k n to R defined by f Q x is f x – Q x. And so again by the rising sun lemma there exist a set E k n inside I k n E k n is a countable union of disjoint intervals I k n j are disjoint open intervals. And if I k n j is equal to this interval I a k n j, b k n j then we have f of b k n j is greater than or equal to the f of a k n j. And for any x which does not belong to this E k n so this is the shadow set in the rising sun lemma for y greater than or equal to x we have f of y is less than or equal to f of x.

So this again should be f q so now let us try top unpack what f this inequality means for f Q.

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$$f_Q(b_{k,n}^j) \geq f_Q(a_{k,n}^j)$$

$$\Leftrightarrow \quad f(b_{k,n}^j) - Q b_{k,n}^j \geq f(a_{k,n}^j) - Q a_{k,n}^j$$

$$\Leftrightarrow \quad f(b_{k,n}^j) - f(a_{k,n}^j) \geq Q \underbrace{m(I_{k,n}^j)}_{(b_{k,n}^j - a_{k,n}^j)}.$$

So f Q b k n j greater than or equal to f Q a k n j this is the same as saying that this definition f of b k n j – Q b k n j is greater than or equal to f of a k n j – Q a k n j. And this is the same as saying that f of b k n j – f of a k n j is greater than or equal to Q times the measure of I k n j which is just b k n j – a k n j.