Measure Theory
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Module No # 14
Lecture No # 73
Riesz's Rising Sun Lemma

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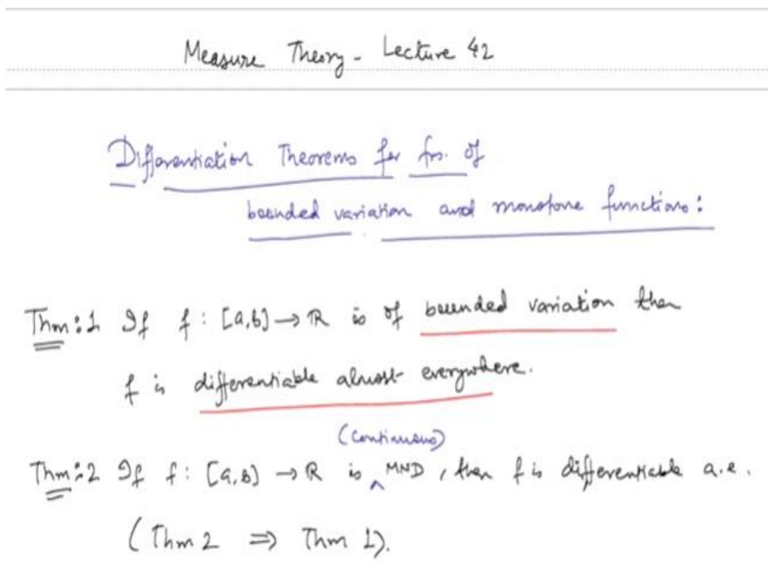

In the last lecture we have seen that function of bounded variation can be expressed as a difference of 2 monotonically non-decreasing bounded functions on this interval a, b. And in the lecture we will see a differentiation theorem for functions of bounded variation and for monotone functions on compact intervals. So of course boundedness for monotone functions is not required as an assumption for compact interval because a monotone functions on a bounded interval will automatically be bounded if it is whether it is monotonically non-decreasing or monotonically non-increasing.

But now we can state the theorem that if, f is a function of bounded variation on a compact interval then f is differentiable almost everywhere. So here we have only assume that it is bounded variation what we will actually show is the following theorem that if, f is monotonically non-decreasing then f is differentiable almost everywhere. So, because the function of bounded variation can be, written as a difference of 2 functions which are monotonically non-decreasing.

Then differentiability almost everywhere for both these functions will imply the differentiability of the function of bounded variation. So theorem let us call this theorem 1 and let us call this theorem 2 so theorem 2 will imply theorem 1. And in fact we will somewhat we can rather strengthen our hypothesis and we will assume that this is a continuous function which is monotonically non-decreasing.

So once we have a established it for continuous function then we will use what is called a continuous single decomposition for monotonically non-decreasing functions and prove it for both these parts of the decomposition one part will be continuous and the other part will be a limit of what are called functions with jump discontinuity or jump functions with they will also be differentiable almost everywhere.

So let us look at the proof of theorem 2 which will imply theorem 1 so this theorem is going to have a rather long proof and they are many lemma's that are needed as the preparatory lemma and we will do this step by step.

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Preparatory Lemma 1: If $f : [a,b] \to \mathbb{R}$ is monotone, then f is measurable. [Ex]

Preparatory Lemma 2: [Riesz's 'Rising Sun' lemma]

Let $f : [a,b] \to \mathbb{R}$ be a continuous fn. and

$$E := \{ x \in [a,b] : \exists\, h = h_x > 0 \text{ s.t. } f(x+h) > f(x) \}$$

Then E is open, and if E is non-empty, it can be written as a countable union of disjoint intervals $\{(a_k, b_k)\}_{k\geq 1}^{\infty}$.

For each $k \geq 1$, either $f(a_k) = f(b_k)$ or if $a_k = a$, $f(b_k) \geq f(a_k)$.

So the first preparatory lemma that we need is that if you have a monotone function on a real value monotone function on a compact interval. Then it is measureable so this is not very difficult to show and I will leave it as an exercise. The second preparatory lemma that we need is a very useful result called the rising sun lemma and it is due to reach. And it says that if f is the function from a, b a compact interval a, b to R so a real valued function which is continuous then if you defined e to the set of points in x.

So, this should be a, b set of points in a, b such that there exist a strictly positive number h depending on x. So h is h x such that f x + h is strictly greater than f of x. Meaning that the value that f takes at x is strictly lower than the value that f takes on a point which is strictly on the right of x. So then they claim says that this set E is an open first of all and if E is non-empty. If it is non-empty it can be written as a countable union of disjoint intervals a k, b k equals to 1 to infinity.

So, just a countable union of intervals of the form a k, b k and E is a disjoint union of all these intervals. And the last part which is the most crucial is that for each k either we have f a k equals f b k or if one of the end points is actually equal to a. Then f b k is greater than or equal to f a k so before we proceed let me give a graphically illustration why it is called the rising sun lemma.

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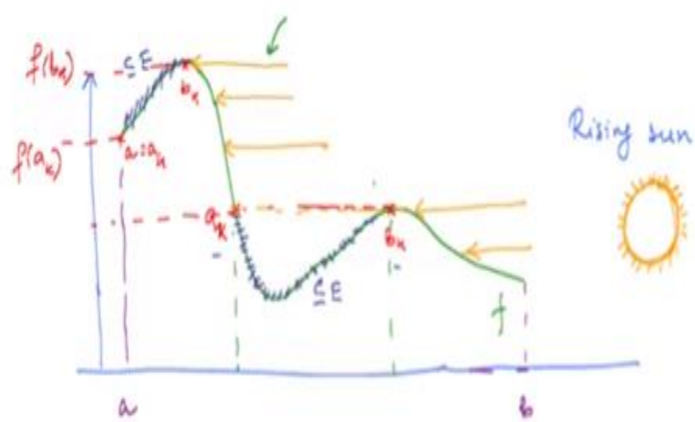

So this picture we have here so on the right hand side we have the rising sun rising from the east. So on the right hand side and so that the Riesz of the sun are horizontally coming from right to left so these are the Riesz of the sun. And if we have this function f in the green line so this is the graph of the function f from a to b. So there will be points where the sun Riesz can hit directly on the graph of the function f, there will be some points where the sun Riesz will hit directly.

And there will some other points where sun Riesz cannot hit because they lie in the shadow area. So here these shaded points or the shaded area where this sun Riesz cannot hit. Because you can think of them as mountains and these are the so called valleys. And so here you can so these are precisely the subs the disjoint union the intervals can make up this set E. So this interval for example from a k to b k this is the sub set of E and here again the subset of E.

Except that when the starting point is itself the end point of one of the opening intervals then you no longer have a k equals b k. So if a k and b k lie in the interior of the, a and b then f of a k will be equal to the f of b k. So this is that f takes the same value on a k and b k except when a k is the starting point a in which case f b k. So this is f b k this is greater than or equal to f a k okay. So this is why it is called the rising sun lemma.

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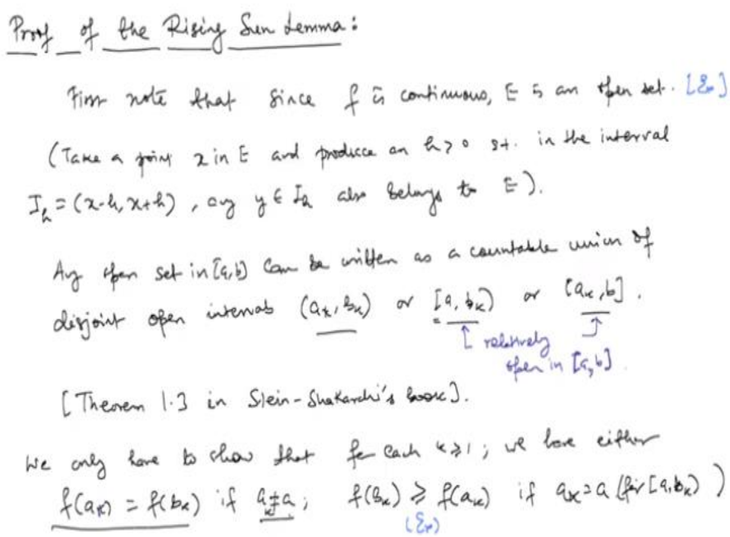

And now let us look at the poof of this lemma so to start the proof of the rising sun lemma first note that since f is continuous function f is continuous E is an open set. So I will leave it as an exercise just by writing definition of continuity and take a point x in E and produce an h positive such that in the interval x – h to x + h any. So let me write this as I h any y in I h also belongs to E. Meaning that there will be given this point y there will be point to the strictly to the right of y on which f takes a higher value okay.

So this is not very hard to show and I will leave it to you as an exercise. And now we have an open set E and any open set in R can be written let me take it a, b can be written as a countable union of disjoint open intervals a k, b k or a b k. So a can be included here or a k b so this can also occur in the disjoint unions and in fact we will also consider these kinds of intervals in our decomposition of e into a countable union of disjoint intervals.

So not just this kinds of intervals but also where you can include a, or include b because these are also relatively open sets. So this set and this set is a relatively open in a, b okay so this is a rather a general topological result that you can consult. For example you can look at theorem 1.3 in Stein and Shakarchi book for a proof of this fact that any open set in a, b can be written as a countable union of disjoint intervals relatively open intervals in a, b.

So now we only have to show that for each k greater than equal to 1 we have either f a k equals f b k if a k is not equal to a and f b k is greater than or equal to f a k if a k equals a. So here in this case the open interval that we consider is of the form a b k excluded. So this is a kind of so this is for the interval a b k so we just let me show this one and I leave this assertion also as an exercise just by looking at the proof that we will see for the equality holds you can try to argue in a similar way to show that this inequality holds. So let us try to show that if a k equals f b k when a k is not equal to a.

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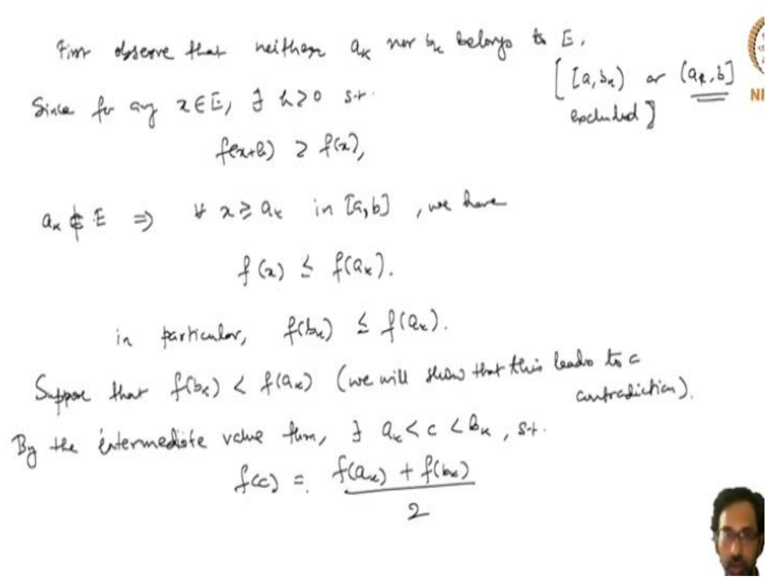

So to show this first observe that neither a k nor b k belongs to E because it is an open interval. So the end points are not included so neither a k not b k belongs to E because these are interior points and we are not considering intervals of the form a b k or a k b okay. So these are excluded so in fact I think you can also consider this one it will be clear from the proof that we will give. So since for any x in E there exist an h greater than 0 such that f x + h is greater than f x.

And because a k does not belong to E this implies that for all x greater than or equal to a k in a b we have f x is less than or equal to f a k. Because otherwise if there was a point where f x was greater than f a k strictly to the right of a k then it would belong to E and we have seen that a k is not in E. So in particular f of b k is less than or equal to f of a k we now suppose that f b k is strictly less than f a k and we will arrive at a contradiction we will show that this leads to a contradiction okay.

So we have to show that f b k is equal to f a k so this will prove our result so because f is continuous by the intermediate value theorem there exist a c which is between a k and b k strictly line between a k and b k such that f of c is equal to f of a k + f of b k over 2. So this is the average values of the function f that it takes on a k and b k and such as c exist by the intermediate value theorem.

Now it could happen that they are infinitely many such points c so what we will do is we will take the farthest to the right the point c which satisfies this we will take it to be farthest possible to the right hand side so largest possible .

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Since the set $f^{-1}(\underbrace{\{f(c)\}}_{\text{closed}})$ is closed, we take.

$$c_0 = \sup\left\{ c \in (a_k, b_k) : f(c) = \frac{f(a_k) + f(b_k)}{2} \right\} \in f^{-1}(f(c)).$$

So, $f(c_0) = \dfrac{f(a_k) + f(b_k)}{2}$

To arrive at a contradiction, we will show, $\exists\, c' > c_0$ and $c' < b_k$

Such that $f(c') = f(c_0) = \dfrac{f(a_k) + f(b_k)}{2}$

First note that since $c_0 \in E$, $\exists$ a point $d > c_0$ s.t. $f(d) > f(c_0)$.

So since the set f, inverse f c is closed this is a closed set so the inverse image under continuous function this is going to be a closed set. We take C naught to be the supremum of point c in a k, b k such that f of c is equal to f of a k + f of b k over 2. And this C naught is going to belong to the set this is included in the set f inverse of, f c because it is a closed set. So in particular we also have f C naught is equal to f a k + f b k over 2.

And now to arrive at a contradiction to arrive at a contradiction we will show that there exist C prime strictly greater than C naught and C prime less than the right end point b k. So b k here is the right end point such that f of C prime is also equal to f C naught this is equal to f a k + f b k over 2 and this is going to be a violation of the fact that C naught is the supremum of such values. So to show this first note that since C naught belongs to E there exists a point d greater than C naught such that f of b is strictly greater than f of C naught.

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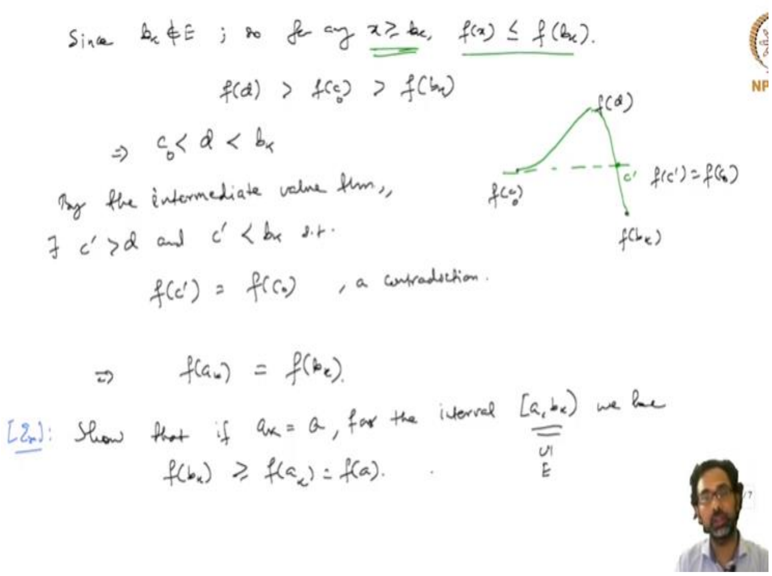

Now we also have that b k does not belong to E so both these end points a k and b k did not belong to E. So for any x greater than equal to b k we have f x is less than equal to f of b k right. This implies that so we had f of d greater than f of c but f of c is again strictly greater than f of b k because it is the average of the values of, f a k and f b k. And this implies that f b is greater than f b k which means that d must be less than b k because for all values greater than or equal to b k we must have less than or equal to f b k.

So we have found a point between c and b k strictly line in the interior of this 2 points such that f d is greater than f c is greater than f b k. So here we have f c here we have f of b k and here we have f of d. So the graph must go to f d and then come back to f b k so by the intermediate value theorem there will be again another point c prime. So here I should take c naught rather than c so this is c naught so there is another point such that f c prime equals f c naught right.

So since f d is so I can just write similarly so by the intermediate value theorem there exist c prime greater than d, and c prime less than b k such that f of c prime equals f of c naught. But this is a contradiction which implies that f of a k must be equal to f of b k. So this proves the rising sun lemma except that we still have to make sure that the inequality holds so as an exercise show that if a k is equal to left end point a for the interval a b k we have f b k is greater than or equal to f a k which is f a.

This should be almost immediate because f must be non-decreasing on this entire interval because all of these points lie in E. So this whole thing is a subset of E so f must be non-increasing so I will just leave it as an exercise for you to show you can argue similarly or you can give another argument to show this inequality. So, this proves the rising sun lemma and now we will use the rising sun lemma to establish our differentiation theorem for monotone continuous functions on a, b.