Measure Theory

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Module No # 14
Lecture No # 72
Differentiation theorems: Almost everywhere differentiability for Monotone and Bounded Variation functions – Part 2

Now let us look at the proof of the first claim and I will only show first that the positive variation is monotonically non-decreasing and I will leave the case for the negative variation to be monotonically non-decreasing as an exercise. So to, show that this positive variation is monotonically non-decreasing.

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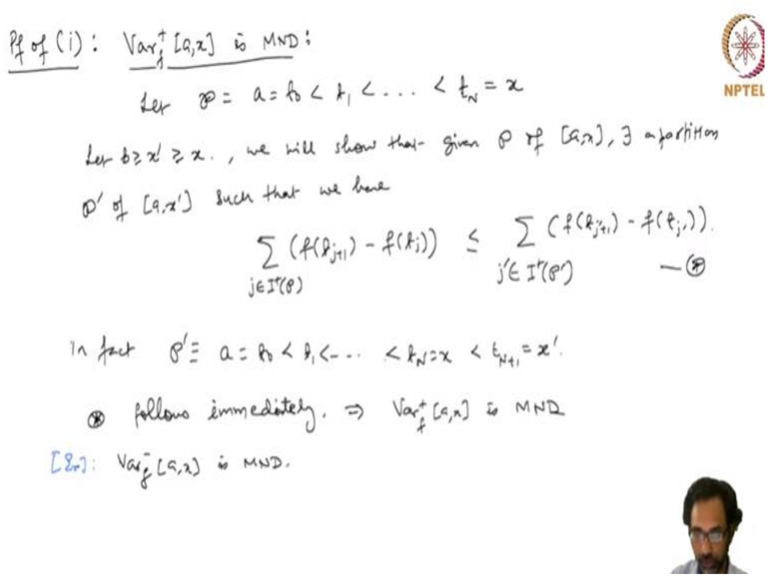

So let p be a partition of this interval a, x so a is t0 then you have points t 1, t 2 up to t n this is equal to x and let also x prime be a point greater than or equal to x. This is still less than equal to b so we will show that given any partition p of a, x there exist a partition p prime of a, x prime. Such that we have that the sum j in I + p of these terms f t j + 1 – f t j is less than or equal to the sum j prime belong to I + of p prime f t j + 1 – f t j prime + 1 – f t j prime.

And in fact it is quite easy to create this partition in fact p prime can be taken to be so start with so it is just an extension of p to this interval a, x prime. So we have all the points that we have already had up to x and then we have t n + 1 this is equal to x prime. So this is just adding one more point in the partition and you have immediately this inequality because the sum on the right will contain the sum on the left.

So this inequality star follows immediately and therefore Var + f a, x is monotonically non-decreasing. So similarly when I prove leave it as an exercise that Var – f a, x is monotonically non-decreasing.

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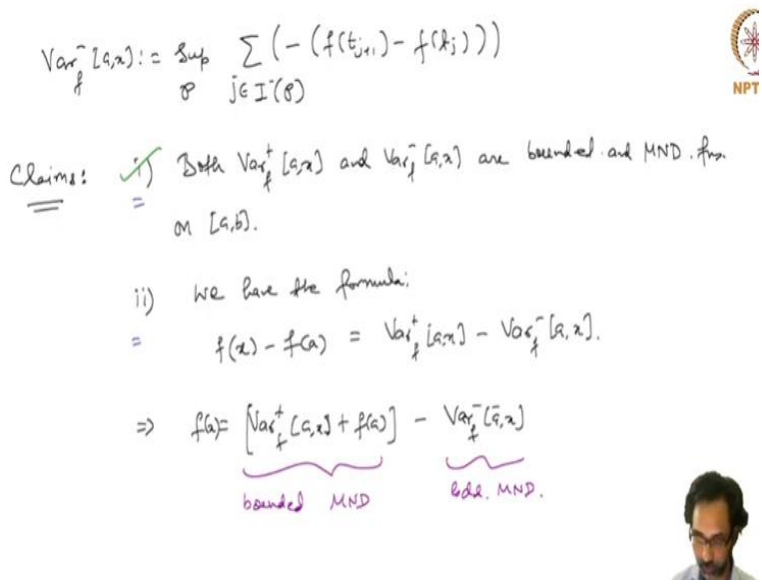

Now to show that these 2 are in fact bounded I will just again use the positive variation as an illustration of the proof is bounded. In fact I am going to show that Var + f a, x is less than or equal to the total variation or the variation of, f over a, x and this is less than or equal to the total variation a, b and this is finite because f is of bounded variation. So we will show this so it is not very difficult to show this again this is about extending partitions so let p be the partition of a, x.

So a = t0, t 1, t 2 and so on and we have t n equals x and we have immediately that j in I + p of this sum f t j + 1 – f t j is less than or equal to the whole sum j = 0 to n – 1 f modulus of, f t j + 1 – f t j. Because again the sum on the right contains all the terms on the left so this means that Var + f a, x is less than or equal to Var f a, x and it is immediate that this is less than or equal to the total variation over a, b which is finite.

So it is also bounded and so we have shown that the positive variation is both bounded and monotonically non-decreasing. And as an exercise again you should show that Var minus is also bounded. So this proves the first part of the claim so let us go back to the claim this proves the first part which says that both these functions Var plus and Var minus are bounded monotonically non-decreasing. And on the other hand we have this formula f, x – f a, should be equal to the difference of these to functions Var plus and Var minus so let us try to show this.

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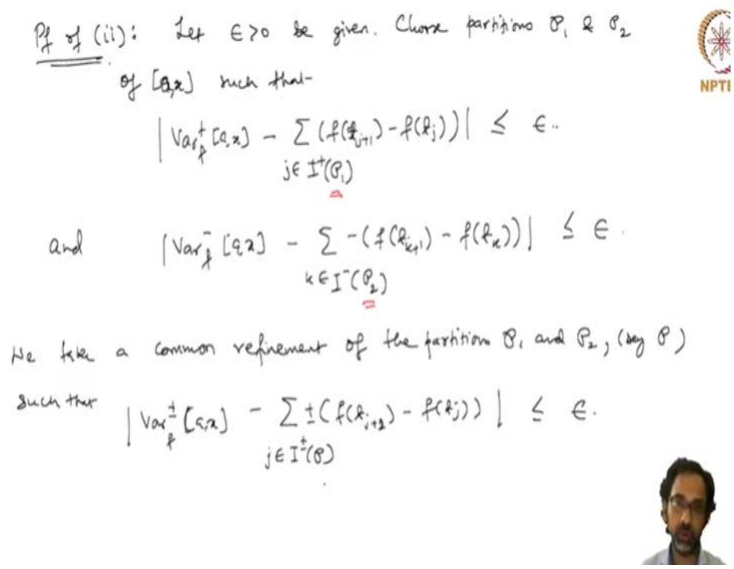

So proof of part 2 to show this choose partitions so first of all let epsilon greater than 0 be given and now we can choose partitions p 1 and p 2 of a, x. Such that we have the modulus of Var + f a, x minus this sum f t j + 1 - f t j in I plus of p 1. This difference is less than or equal to epsilon and similarly the negative variation we can take let us take the different index let us say k in I – P 2. The point here is that when you are using the supremum definition of the positive and negative variations which is the supremum over all partitions.

Then for a chosen epsilon the 2 partitions that you get when you get closer and closer to the supremum may differ when you take the positive variation and the negative variation. So here p 1 might be different from the partition p2. So here we have for the negative variation minus of, f t j – 1 t k + 1 – f of t k. This is also less than or equal to epsilon. And now we can take a common partition common refinement of the partitions of p 1 and p 2.

So let us call this common refinement say p such that on this common refinement we still have that Var plus minus f. So I am just putting both inequalities in 1 formula j in I plus minus p now this is the sum of plus minus f t j + 1 – f t j this is less than or equal to epsilon. So this passing to the common refinement is a little bit tricky because the set of indices let us say for the positive variation. When you pass to the common refinement then this index set I + p 1 might be different from I + p and similarly I – p 2 might be different from I – p

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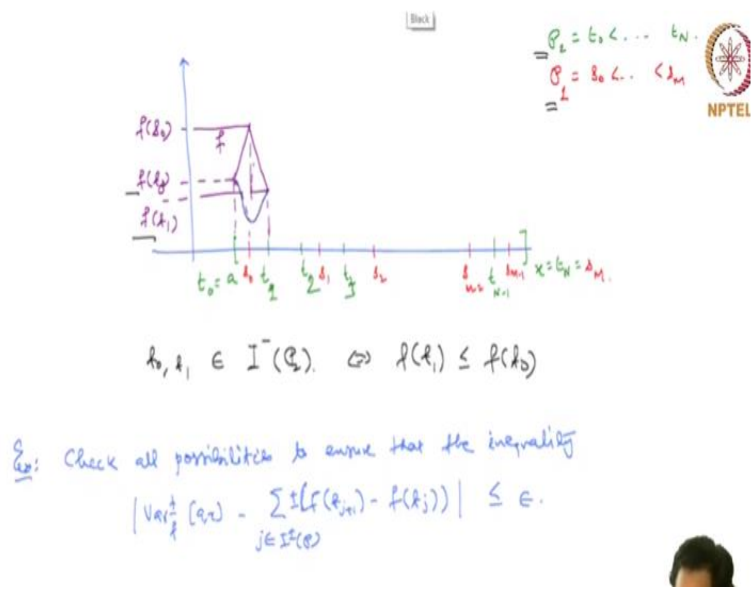

So let us see a graphical representation of the changes in the index set I + and I – that can occur here. I have denoted in the green colored points the partition p 2 that we used for the negative variation. And in the red color points the partition p1 that we use for the positive variation. And we suppose that t0 and t1 belong to I – p 2 which means that so f t1 is less than or equal to f t 0. So this is the situation here f t 1 is less than or equal to f t 0.

But when you plug in the partitioning from t 1 which was for the positive variation then it could happen that t 0 is no longer in I minus because at s 0. So the partition now is t0 is 0t1 and so on so according to the increasing order of the points and here we have a f of t 0 is less than f of s 0 and therefore t 0 is no longer in a I minus but it is in I plus of the new partition P. But we will still have that this variation will still be close to the for the sums given by I+ p and I – p, for both positive and negative variations then this sums will be at most epsilon away from the positive and negative variation.

So I will leave it as an exercise again check all possibilities to ensure that the inequality Var plus minus f a, x – j in I plus minus p a modulus of or rather plus minus f t j + 1 – f t j is less than or equal to epsilon. So even though we can have changes in the index set we will still have this inequality. So once we have this then it is not very difficult to show this f x – f a, is also close to the difference Var plus and Var minus by 2 epsilon.

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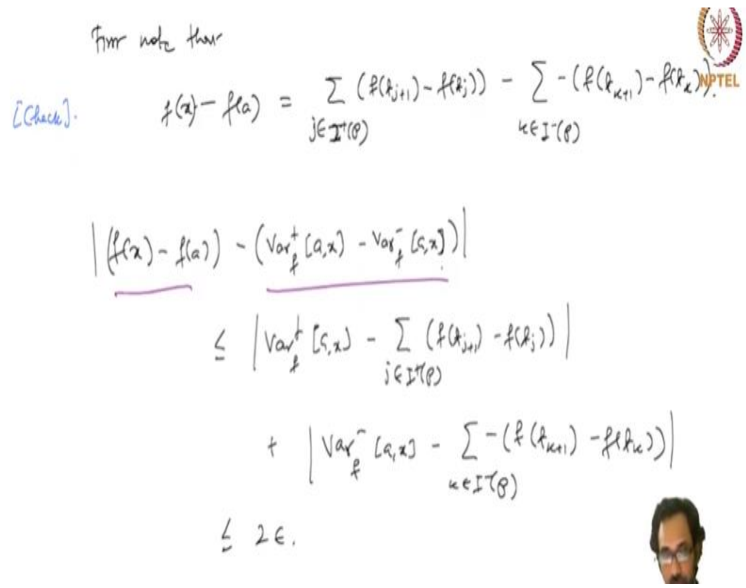

So let us see first that note that f x – f a is equal to the sum j over I + p f t j + 1 – f t j – k in I – p – f t k + 1 – f t k. So this is simply by using the sort of the telescoping sum which is broken p into 2 pieces 1 for the positive index set and 1 for the negative index set. So check these formula holds and secondly we have that f mod of, f, x – f, a – Var + f a, x – Var – f a x. This is now less than or equal to the sum so this is Var + f a x – j in I p f t j + 1 – f t j + Var - f a x - k in I – p – f t k + 1 – f t k and both these terms are less than or equal to epsilon.

So this less than or equal to 2 epsilon and since epsilon was arbitrary we have equality of these 2 terms. So this shows that f is a difference of 2 bounded monotonically non-decreasing functions. So 1 is given by Var + f a x = f a and the other one is given by Var – f. So this proves the statement of the lemma.