Measure Theory

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Module No # 14
Lecture No # 71
Differentiation theorems: Almost everywhere differentiability for Monotone and Bounded Variation functions – Part I

In the last weeks lectures we have seen Lebesgue differentiation theorem which showed that the derivative is informally a left inverse for the integral for L1 functions. And in this week's lectures we will be last topic for this course we will talk about differentiation theorem which will established almost everywhere differentiability for some more classes of functions.

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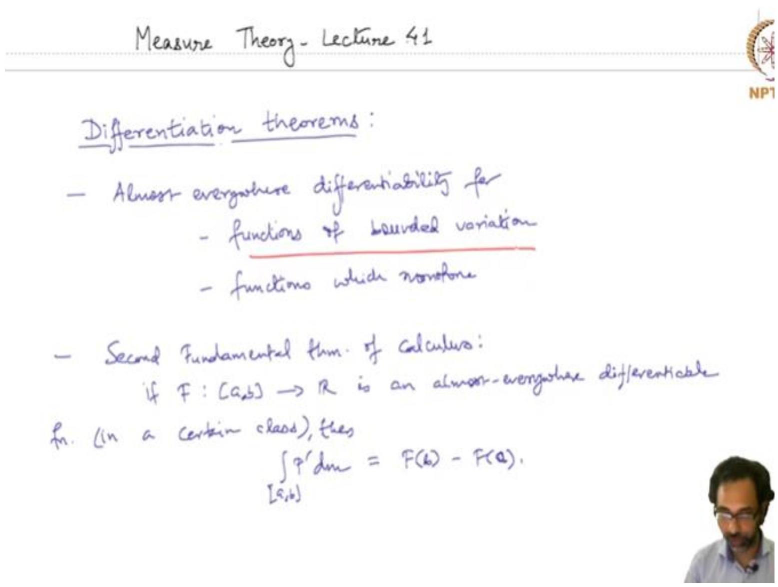

So we will see almost everywhere differentiability for functions of what are called bounded variations functions of bounded variation. And as well as functions which are monotone so either monotonically non-increasing or monotonically non-decreasing. So this differentiation theorems allow us to take the derivative of functions which are not differentiable everywhere but they are still differentiable almost everywhere and we can do analysis on these kinds classes of functions.

We will also touch but rather briefly what is called as the second fundamental theorem of calculus. Second fundamental theorem of calculus which says that if F from a finite interval to R is an almost everywhere differentiable function everywhere differentiable function in a certain class of function. So it is not true for all kinds of almost everywhere differential functions but in a certain class.

Then the integral of the derivative F prime of F d, n this is equal to this is equal to F of b – F of a. Let me just remark here that the goal of treating this topic is not to give you detailed proofs because usually the proofs will be quite long. So I will only give sketches of proofs leave either exercises for you to do yourself or when it is something complicated, then I will give you right reference for you to go and look it up in a book. So let us come to the definition of functions of bounded variation which we will treat first.

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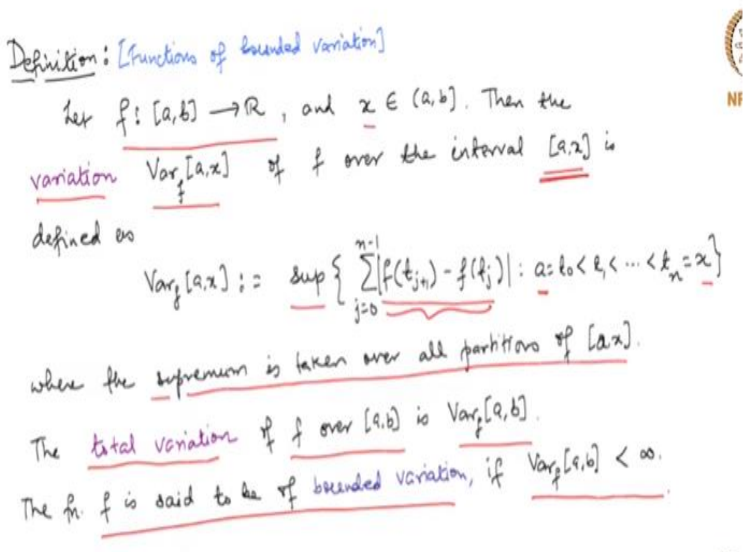

So here is the definition of the functions of bounded variation. So if you consider a function on a finite interval a, b a real valued function and pick a point x between a and b a excluded and b included. Then we define the variation which we denote by Var F a, x of, f over the interval a, x. So this is variation of, f over the interval a, x closed interval and it is defined as the supremum of all this sums.

So the sum of these are the how much f is varying over n points of partitions n points of sub intervals given by a partitioning of the interval a, x. So you take a partitioning t0, t1, t n of this closed interval a, x and then you can take you can sum over all this differences of the function f over it is over the end points of the subinterval. And then you take the supremum overall such partitions and this gives you the quantity called the variation of, f over this interval a, x.

So here note again that supremum is taken over all partitions of a, x and now the total variation of, f over the whole interval a, b is simply defined to be Var f a, b. So again you take the supremum over all partitions of a, b and you take sums of this forms f t j + 1 – f t j and you sum it over the n points of your partition. Finally the function f is set to be of bounded variation f the total variation is finite. So this is a definition of a function of bounded variation.

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Examples: i) If $f : [a,b] \to \mathbb{R}$ is a bounded, monotonically non-decreasing, then f is of bounded variation.

Take any partition $a = t_0 < t_1 < \ldots < t_n = b$.

$$\sum_{j=0}^{n-1} \underbrace{|f(t_{j+1}) - f(t_j)|}_{\geq 0} = \sum_{j=0}^{n-1} (f(t_{j+1}) - f(t_j))$$

↑ telescoping sum.

$$= f(b) - f(a) \leq 2 . \underbrace{\sup_{x \in [a,b]} |f(x)|}_{\text{does not depend on the partition.}}$$

$\Rightarrow$ f is of bounded variation.

So now let us look at a couple of examples of functions of bounded variations so first one is if f from a, b to R is a bounded function, as well as we assume that it is monotonically non-decreasing then f is of bounded variation. So in fact you can take any partition of a, b. So let us say a equals t naught and t1 and so on and t n equals b. So if you take any partition of a, b then this sum j = 0 to n – 1 of modulus of, f t j + 1 – f t j.

Since f is monotonically non-decreasing this is greater than equal to 0 and so you can drop the modulus sign so this is nothing but j = 0 to n – 1 f t j+1 – f t j. But this is now a telescoping sum and so in the end you will only get f b – f a. And since f is also bounded this is less than or equal to 2 times the supremum of mod f x in a, b. So we see that you can take any partition and it will be bounded by this value which does not depend on the partition.

And so we can take the supremum on the left hand side which will give you that f is of bounded variation because the total variation will be finite. So this is our first example and we will see that in some sense all bounded variation functions can be described using monotonically non-decreasing functions.

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(ii) If $f : [a,b] \to \mathbb{R}$, is Lipschitz : $\exists\, M > 0$, s.t.

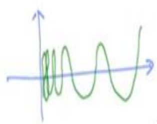
$$|f(x) - f(y)| \le M \cdot |x-y| \quad \forall\, x, y \in [a,b].$$

Then f is of bdd. variation [Ex].

Non-example: Let $f : [0,1] \to \mathbb{R}$

topologist's sine curve

$$f(x) = \begin{cases} \underbrace{x \sin\left(\frac{1}{x}\right)}_{\text{high oscillations}} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$



f is continuous, but not of bounded variation.

[Ex: Prove this; show that $x^a \sin\left(\frac{1}{x^b}\right)$ is of bdd. variation if and only if $a > b > 0$]

Now another example is if, f from a, to b same domain is a Lipschitz map which means that there exist a constant m such that the modulus of, f x – f y is less than or equal to m times modulus of x – y for all x, y in a, b. Then f is of bounded variation and again one can use a similar argument that we used in the first example and use a telescopic sum to prove this. And I will leave it as an exercise to show that this is of bounded variation.

Now let us look at a non-example so a function which is not of bounded variation in fact this function is also to be continuous function. So let f from 0, 1 to R given by f x equal x sin 1 over x if x belongs to 0, 1 and 0 if x = 0. So this is also sometimes called the topologist sign curve and in fact this is the continuous function. So f is continuous but not of bounded variation so I leave it to an exercise to show this the point here is that this factor sin 1 over x as a lot of high oscillations.

And which means that when x goes closer and closer to 0 there is a lot of wiggle in the graph of this function f x. So the graph of the function f x looks something like this so when you go closer and closer to 0. So there is a lot of wiggle near 0 but still the function f is continuous and the limit as f x goes to 0 from the right hand side is precisely 0. Even though it is not well defined x sin 1 over x is not well defined at 0. But still the limit exists and this makes the function continuous.

But this oscillatory nature of sin1 over x breaks the bounded variation because there are infinitely many ups and downs in this curve as x goes to 0. So I like you to make the argument precise prove this and in fact show that x to the power a sin 1 over x to the power b is of bounded variation. If and only if a, is greater than b and I should add here that both are taken positive. So it is a bounded variation if and only if a, is greater than strictly greater than b and both are positive.

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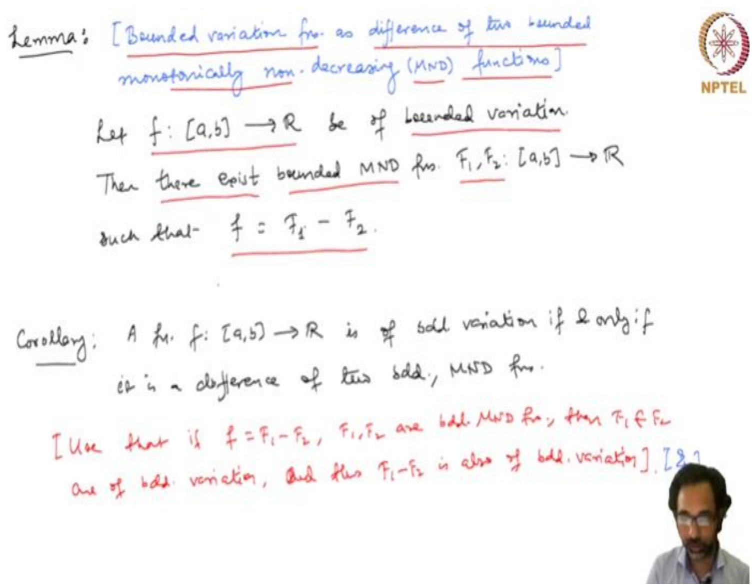

Now we come to the first important property of functions of bounded variations and this is given by this lemma and which says that bounded variation functions can be expressed as a difference of 2 bounded monotonically non-decreasing functions which can be defined from the given function of bounded variation. So I am going to use this abbreviation MND for monotonically non-decreasing because we are going to use it very often.

So the statement of this lemma is as follows so if f is a function of bounded variation on this interval a, b then there exist bounded monotonically non-decreasing functions f 1 and f 2 such that f is the difference of, f 1 and f 2. Of course as a corollary we have that a function f is of bounded variation if and only if it is a difference of 2 bounded monotonically non-decreasing functions. Because the only if conditions is given by this lemma and the, if condition follows the lemma and the, if condition follows.

Because our first example shows that any bounded monotonically non-decreasing function is of bounded variation. So if, f is a difference of 2 bounded monotonically non-decreasing functions f 1 and f 2 then both f 1 and f 2 are of bounded variation and it is not very difficult to show usual triangle inequality that the sum of 2 bounded variation function is again of bounded variation. So here use that if f equals f 1 – f 2 where f 1 f 2 are bounded monotonically non-decreasing functions.

Then both f 1 and f 2 are of bounded variation and thus f 1 – f 2 is also of bounded variation. So the total variation of f 1 – f 2 can be estimated by the total variation of f 1 and the total variation of f 2. So I will leave it as an exercise again for you to thresh out the proof for this corollary. But now let us look at this a proof of this lemma.

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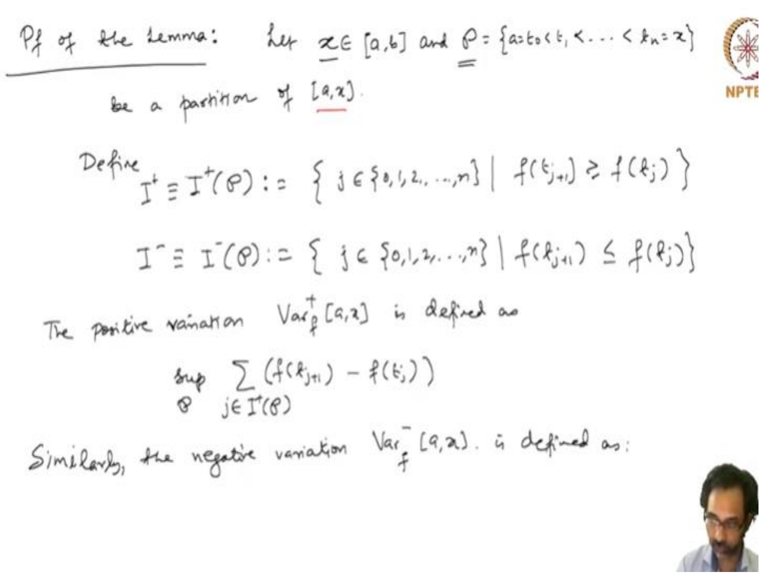

So for this let us fix a point x in a, b and we also fix a partition p given by t naught, t1 and t n of this interval a, x. So this is a partition of this interval a, x now we define 2 sets I + p so I will define denote it as I + when the partition is evident. So this is the set of all j in 0, 1 to up to n such that f t j + 1 – well f t j + 1 is greater than or equal to f t j. So these are the sub-intervals of partitions on which the right most end point the f as higher value on the right most and end point than the left end point.

And similarly I – which is I – p this is the connection of all this indices j such that f t j + 1 is less than or equal to f t j. So which means that f takes lower values on the right end point than the left end points okay. So now we define the positive variation which we denote by Var + f a, x so this is defined as the sum over all partitions of sums taking over the positive I +. S these are the indices where you give get a positive value for the difference f t j + 1 – f t j.

So here you can write f t j + 1 – f t j and so this supremum is taken over all these partitions of the interval a, x. Similarly we can define the negative variation denoted by Var – a, x of, f.

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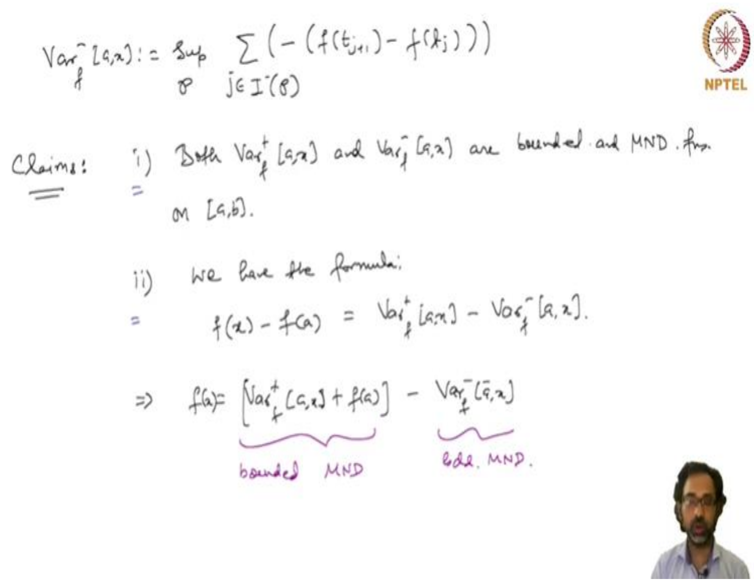

And this defined as with a similar formula so you get j in I minus of p – f t j + 1 – f t j. So here there is a minus sign for each of the terms in the sum so this is Var minus of, f a, x and this one was Var plus of, f a, x. So now I claim 2 things claims that first is that both Var + f a, x and Var – f a, x are bounded and monotonically non-decreasing functions on a, b. So this is the first claim and the second claim is that we have the formula f x – f a is equal to Var + f a x – Var f – a x.

So here so this is f so here of course if assuming both these claims are true then we immediately get the statement of this lemma because this implies that f is equal to Var + so f x = Var + a x + f a. So this one function minus Var – f a x and both these functions are bounded and monotonically non-decreasing. So then we would have expressed our function which was of bounded variation as a difference of bounded monotonically non-decreasing functions.

So it is we only left to show these 2 claims where first is that both are bounded and monotonically non-decreasing on a, b.