Measure Theory

Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science – Madras

Lecture – 70
Lebesgue's Differentiation Theorem Statement and Proof - Part 2

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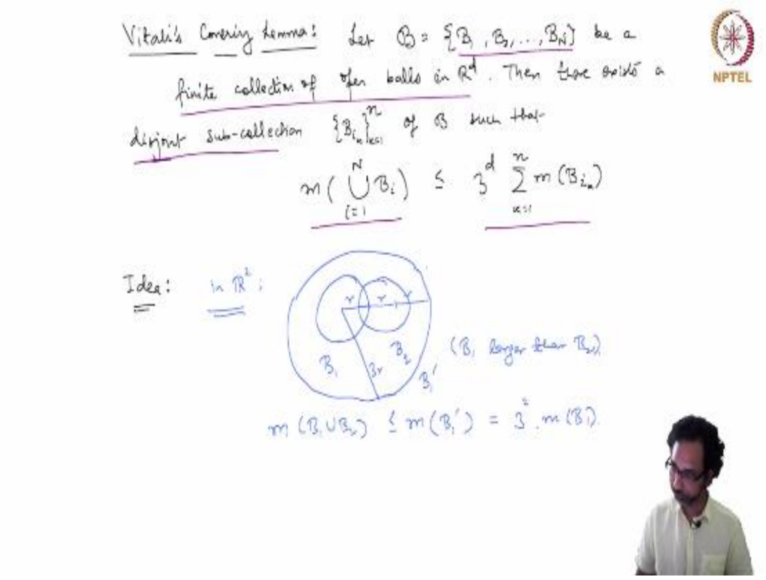

So let us look at Vitali's covering lemma is the following statement that if you have a finite collection of open balls in R d say B 1, B 2 up to B N then there exist a disjoint sub-collection so the disjointness is important here. Disjoint sub-collection B i k from 1 to small n of B such that the measure of the union is bounded above by 3 to the power d times the sum of this balls from the disjoint sub-collection.

So, the idea here is quite simple. It simply says that if you have two circles B 1 and B 2 then the second circle say B 1 is larger than B 2 then if you take the circle or the ball of radius three times the radius of B 1 so one here, one here and one here. So three times the radius r r r then this will envelop both B 1 and B 2. So, in this case we only have two open balls B 1 and B 2 and the union of B 1 and B 2 can be enveloped by a ball of center the same center as B 1 and three times the radius.

So that here you have m of B 1 union B 2 is less than or equal to the measure of the ball. So, let me write this as B 1 prime which has radius 3 r so this is less than or equal to the measure of the ball B 1 prime and the measure of the ball B 1 prime is precisely 3 square times the measure of B 1. So this is in R 2 we have this picture. So, the Vitali's covering lemma is a generalization of this statement for any finite collection of open balls in R d.

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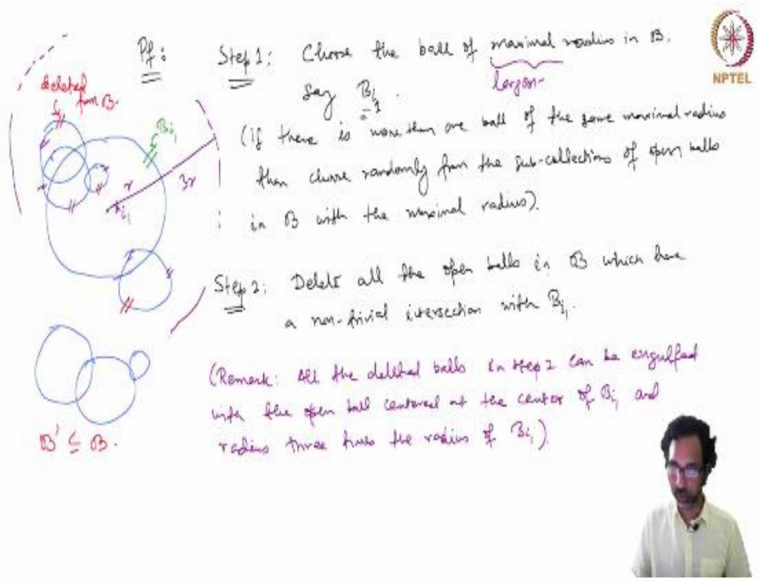

So, let us here proof which is not constructive but rather more algorithmic? So, step 1 choose the ball of maximal radius among this collection in B. So say B i 1. So here note that if there are more than one ball if there is more than one ball of the same maximal radius then we can choose randomly from this sub-collection then choose randomly from the sub-collections of open balls in B with the largest maximal radius.

So, maximal radius here means the largest radius with a maximal radius. So, if there are more than one then we can just pick one and then we repeat this process for the next one in particular order and because they are finitely many there are no issues with choices. So, the first step is to choose the ball of maximal radius in B and say this is B i 1. So, step 2 delete all the open balls in B which have a nontrivial intersection with B i 1.

So what are we doing here so suppose we have many balls like this something like this and so if you choose the one with a maximal radius this is this one this is the one with a maximal radius which is denoted B i 1 and now we are going to delete all the open balls in B which have a non trivial intersection with B i 1. So all these little ones which are either inside or have a nontrivial intersection with them with the B i 1 are deleting.

So these are all deleted from B and now we are left with a sub-collection B prime in B and then we are going to repeat the same process. Now as a remark note that all the deleted balls in step 2 can be engulfed with the open ball centered at the center of B i 1 so with the same center of B i 1 say x i 1 and radius three times the radius of B i 1. So because if you just as we saw before in this example any two balls which intersect if you take the larger size ball and you take the ball of radius three times its radius then it will engulfed both these balls.

So in fact if you have any number of finite number of open balls this can be done for the same argument applies and so this all this deleted balls this one, this one, this one, this one and this one they are engulfed by another big ball which has radius thrice the radius of B 1. So this is r, this is 3 r and you can engulf all these balls with the bigger ball and now we are left with a sub-collection B prime for which we repeat.

So we get another largest another ball of maximal radius B i 2 then we delete in the next step all the open balls intersecting B i 2 and so on and since they are only a finite number of balls this repetition of step 1 and step 2 will terminate in finite time.

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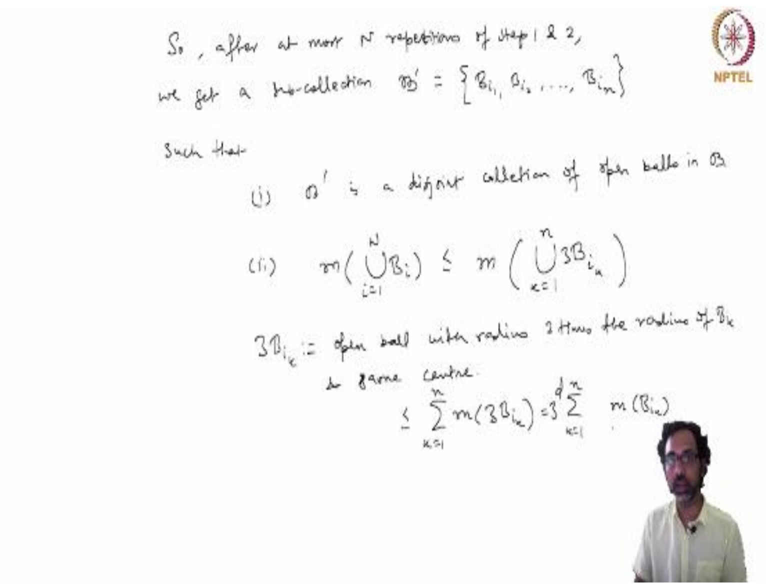

So after at most N repetitions of steps 1 and 2 we get sub-collection B prime given by B i 1, B i 2 up to B i n such that first is that this is a disjoint collection B prime is a disjoint collection of open balls in B because we have deleted at each point all the balls intersecting B i 1. So, therefore in particular none of B i 2 up to B i n intersect B i 1 and similarly for the rest. So this is a disjoint collection of open balls in B and secondly we have that the measure of the union B i i from 1 to N.

This is less than or equal to 3 to the power d times well let me write this first this is less than or equal to the union k = 1 to n the ball with the same radius as B i 1, but with three times B i k the three times the radius. So three B i k is the open ball with radius three times the radius of B i k and same center and now again using the scaling property for the measures of balls this is less than or equal to the sum bounded above by the sum of the measures of 3 B i k and this is precisely 3 to the power d measure of B i k k = 1 to n.

And this is what we wanted because we can take the 3 to the power d outside. So we get disjoint collection of open balls in B which satisfies this inequality that we wanted so this is the proof of Vitali's covering lemma and now let us see how the Vitali's covering lemma will help us in proving the Hardy-Littlewood Maximal inequality in R d.

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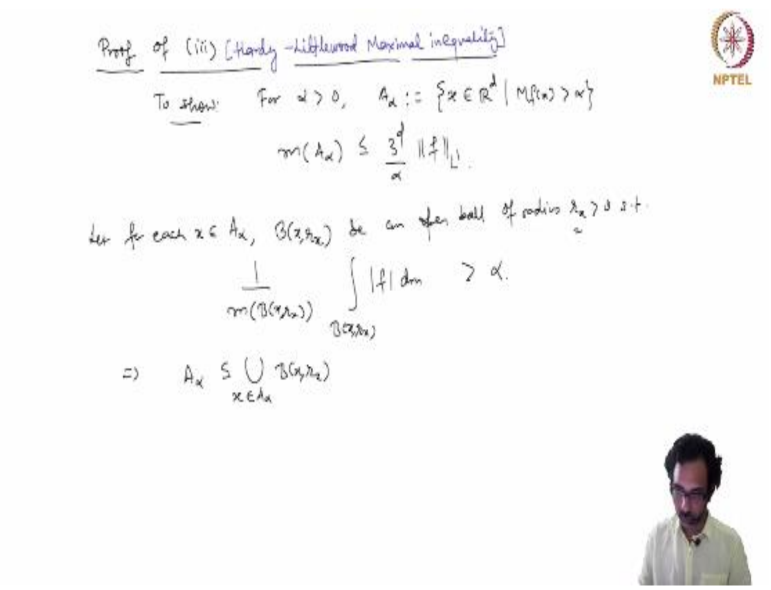

So, we return to the proof of part 3 which is the Hardy-Littlewood Maximal inequality. So, you have to show that for alpha positive the set A alpha defined as the set of all point x such that M f of x is greater than alpha as this satisfies this bound which is the measure of A alpha is less than or equal to 3 to the power d over alpha times the L 1 norm of f. So, remember that f that we started with was L 1 absolutely integrable function.

So, let for each x in A alpha B x be or rather B x r x be an open ball of radius r x positive such that we have 1 over measure of B x r x integral over B x r x mod f d m is greater than alpha because if x is an A alpha you can always find such a radius r x and now this is a collection of open balls covering A alpha. So, this implies that A alpha is equal to x in A alpha B x r x is covered by these open balls. So now if we take so I am going to use.

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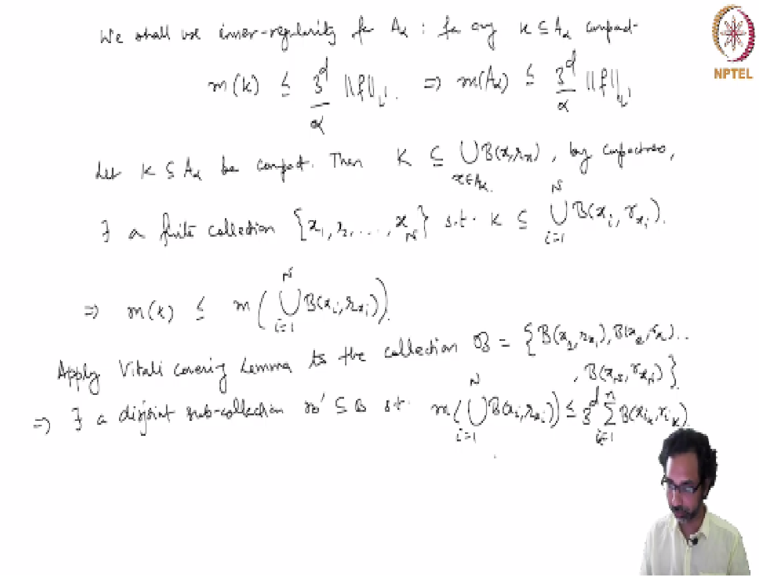

We shall use inner regularity for the set A alpha so which is Lebesgue-measurable set in particular we showed that this is an open set. So, in particular it is Lebesgue-measurable and so inner regularity holds and so we will show that for any compact k inside A alpha we have the required bound which is 3 to the power d over alpha norm of L 1 and by inner regularity this will follow that this would imply that measure of A alpha itself is less than or equal to 3 to the power d over alpha L 1 norm of f by taking the supremum on the left hand side over all compacts inside A alpha.

So, let us take so let k be a compact subset of A alpha then k is covered again by the union of all these balls B x r x and by compactness there exist a finite collection let us say x 1, x 2 so these are the centers of these balls x n. Let me write capital N here such that k is covered by the union i from 1 to capital N B x i r x i. So this is now finite collection of balls so now we have measure of k is less than or equal to the sum or rather the measure of the union i = 1 to N B x i r x i.

Now we are going to apply the Vitali covering lemma to the collection B given by this sets B x 1 r x 1 B x 2 r x 2 and so on B x N r x N. So then we can extract this implies that there exist a sub-collection a disjoint sub-collection B prime of B such that the measure of the union i = 1 to N B x i r x i is less than or equal to 3 to the power d sum over k = 1 to small n B x i k r x i k or let us just say r i k. So, this was the statement of Vitali’s covering lemma which gave you a bound of the measure of the union of finite many open balls by the sum.

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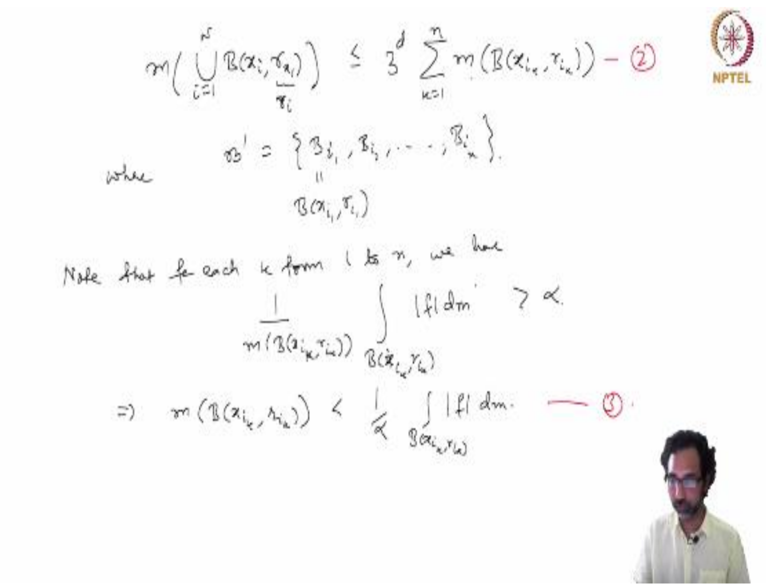

So, let me write this in a new page so this is wrong such that we have the measure of the union i equal to 1 to capital N B x i r x i so let just write this r i for short this is less than or equal to 3 to the power d sum k = 1 to n B x i k r i k where B prime this sub-collection B prime is this collection B i 1, B i 2 or rather so each B i k is B x i 1 r i 1 and so on B i n. So, we now can estimate this measure of the compact set by the Vitali covering lemma here as 3 to the power d i = 1 to n measure of B x i k, r i k and now we will try to conclude.

So, note that for each k from 1 to small n we have 1 over measure of B x i 1 i k r i k integral over B i x i k r i k mod f d m is greater than alpha because this was our choice of the open balls for which this held this inequality held. So, in particular for this sub-collection also this will hold and so this implies that the measure of B x i k r i k is less than 1 over alpha integral B x i k r i k mod f d m.

And so we can use this here let us say this is 3 this is 2 and so we can plug in the values from 3 and 2 so we get.

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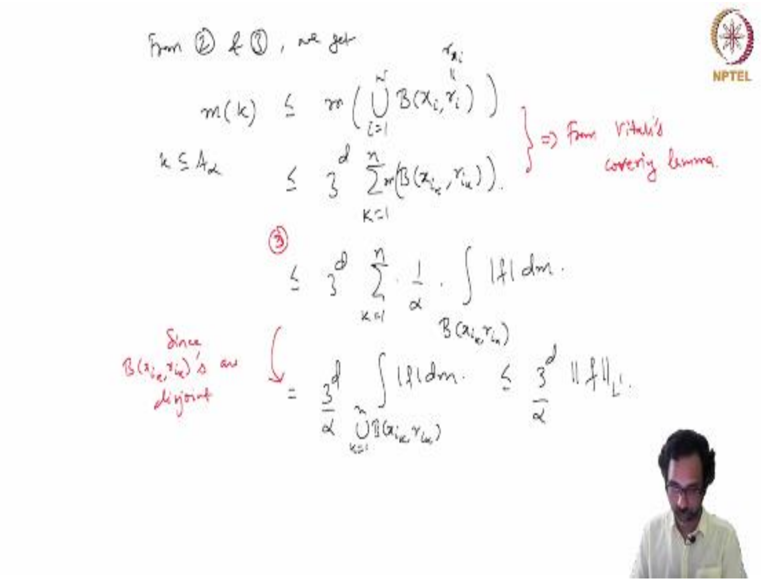

So from 2 and 3 we get that the measure of k which was bounded above by this sum first of all it was bounded above by the measure of the finite union of balls i = 1 to N B x i r i so this was r x i and this was bounded above by the Vitali covering lemma by a sub-collection k = 1 to n measure of B x i k r i k and now from the third so this is from the Vitali's covering lemma and now from the third inequality we get that this is less than or equal to 3 to the power d sum k = 1 to n 1 over alpha times integral over B x i k r i k mod f d m.

And this is equal to 3 to the power d over alpha integral of the union of all these B x i k r i k k from 1 to n mod f d m. Since these are disjoint so this was from the Vitali's covering lemma gave us a disjoint sub-collection. So, this is because since B x i k r i k these balls are disjoint and now this is less than or equal to 3 to the power d over alpha over the integral over all of our d mod f d m and this is nothing, but the L 1 norm of f.

So this is what we wanted to prove that any compact set k satisfies this inequality. So k was a subset of A alpha and now by inner regularity we get the result. So, this completes the proof of Hardy–Littlewood Maximal inequality in R d and now we would like to go back to our proof for the Lebesgue differentiation theorem which we stated only for the dimension one and what we wanted there was this bound on this maximal function which was here.

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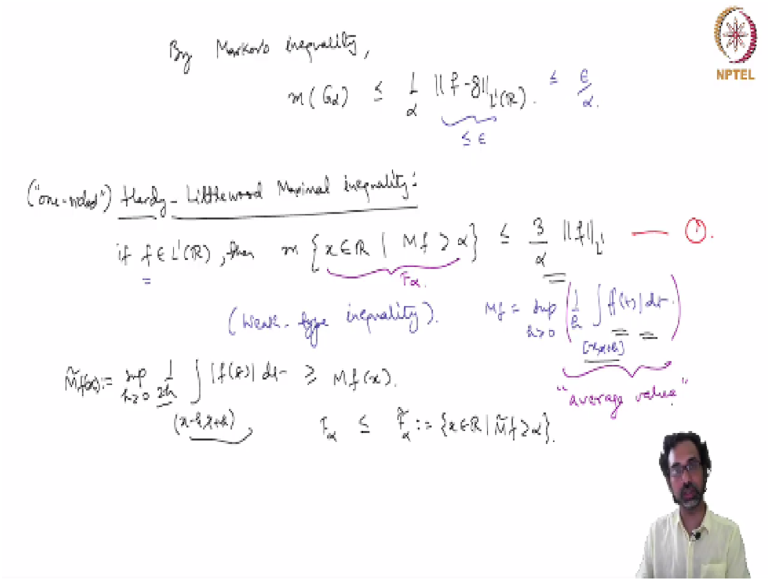

So, we wanted a bound on this maximal function or rather the we wanted the bound on the measure of the set where the maximal function is greater than alpha and our maximal function was slightly different in one dimensional case because it is only from one sided. So it is only for positive values of x so it is what is called a one sided this is one sided Hardy–Littlewood Maximal inequality.

But of course if you have the two sided Hardy–Littlewood maximal inequality which contains an interval of radius let us say 2 h or with the center x then the one sided Hardy–Littlewood inequality will follow because if you define M tilde f so remember that M f is this guy here and now M tilde f if you define it as the supremum so this is at x supremum over h greater than 0 1 over h integral x – h / 2 x + h / 2 or rather let me just take here 2 h and take x – h and x + h and then you take the modulus of f t an integrated over this interval.

So, our Euclidean balls of radius r are now replaced by this kind of intervals. You can take open if you want does not matter here and now we have this is the measure of this interval and we have proved the maximal inequality for this M tilde rather than M because we have used the measures of balls centered at x and with radius r. So, one can now show that we can deduce our result for M from our result of M tilde because this is always greater than or equal to M f x.

Because if you restrict this integral on this interval x to x + h then the integral will be dominated by this integral sorry this integral will be dominated by the integral in M tilde and we will have that F alpha is a subset of F tilde alpha where F tilde alpha is defined using M tilde rather than M and so the measure of F alpha is less than or equal to measure of F tilde alpha and for F tilde alpha we have proved the Hardy–Littlewood maximal inequality.

So, this concludes our proof of Lebesgue differentiation theorem in R and in fact since we have proved Hardy–Littlewood maximal inequality in R d and we have also shown this Vitali covering lemma in R d. So in fact the Lebesgue differentiation theorem can now be easily proved for R d.

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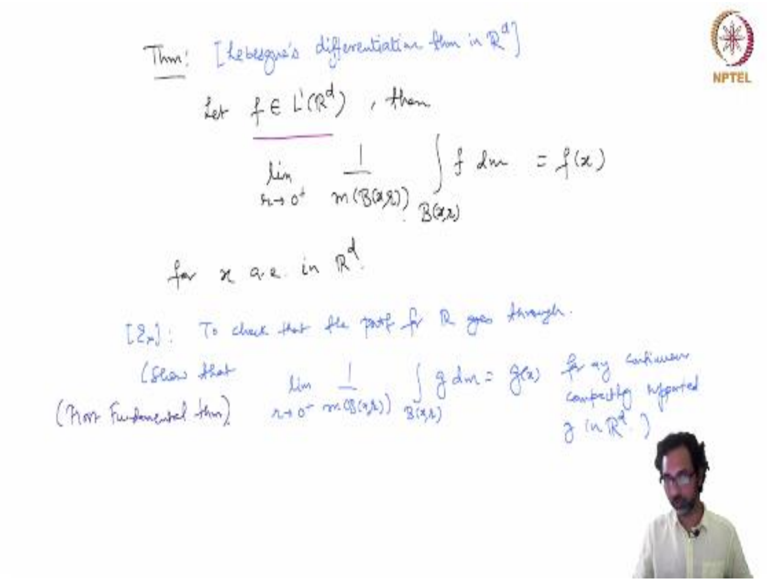

So the statement of Lebesgue differentiation theorem in R d is the following that if you take L 1 function in R d and if you take the limit as R goes to 0 + 1 over measure of B x r integral of f d m over B x r. This is equal to f x for x almost everywhere in R d. So, here as an exercise I will leave it to you as an exercise to check that the proof for R goes through in this case as well.

So, in particular we have to show that the limit as r goes to 0 + 1 over the measure of B x r integral over B x r g d m is equal to g x for any continuous compactly supported g in R d. So, this is the first fundamental theorem of calculus for R d fundamental theorem and I will leave it to you an exercise to prove this result simply by using the continuity of g as we have done for the one dimensional case.

So this completes the proof of Lebesgue differentiation theorem in R d just be repeating all the steps by using the Hardy-Littlewood maximal inequality in R d and then using the Markov's inequality as well as the Hardy-Littlewood inequality to prove that the measure of the sets for which this is not true is equal to 0 by taking the lim sup and the same way that we did for R.

So, this is one of the main differentiation theorems that we wanted to see and in the next lecture we will move to a related topic.