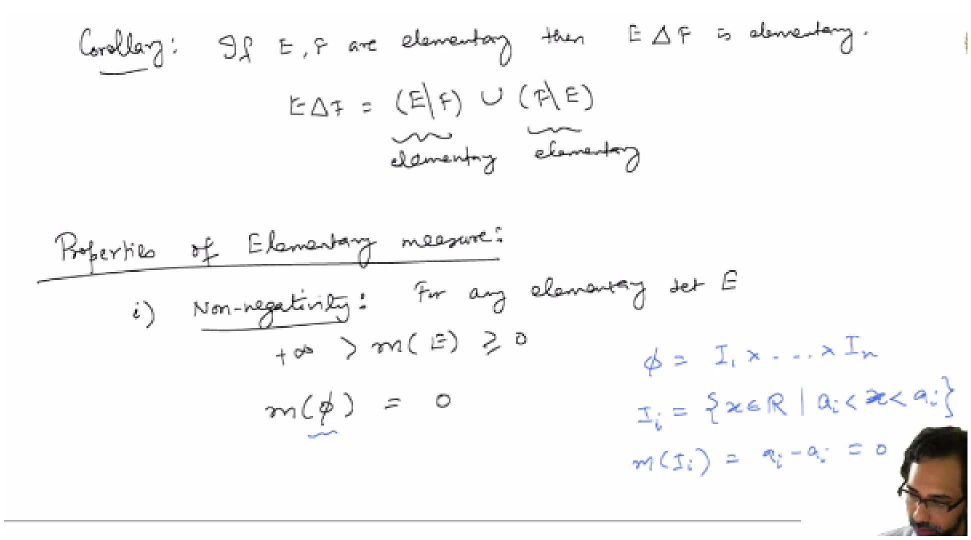
MEASURE THEORY

PROF. INDRAVA ROY

Lecture – 7 Properties of the elementary measure Part 2



Now as an immediate corollary, we have that if E and F are elementary sets in $\mathbf{R}^n$, then the symmetric difference of E and F is also elementary. So, what is the symmetric difference? This is simply $(E–F) \cup (F–E)$ and we know that this is an elementary set and so the union is also an elementary set. Now, let us look at the properties of our elementary measure.

The first property is non-negativity. It says that for any elementary set E, $m(E)$ is a non-negative number and it is always finite. So, it is always a finite non-negative number and if you have the empty set, then the elementary measure is 0. As you can write here $\emptyset = I_1 \times \cdots \times I_n$, to be the Cartesian product of intervals $I_1, \ldots, I_n$, where each I_i is an empty interval. Empty interval meaning that it can be written as$\{x \in \mathbf{R} \mid a_i < x < a_i\}$ as our conventions allow us to have empty intervals like this. Then the measure of $m(I_i) = 0$. So we have that the measure of the empty set is always 0.

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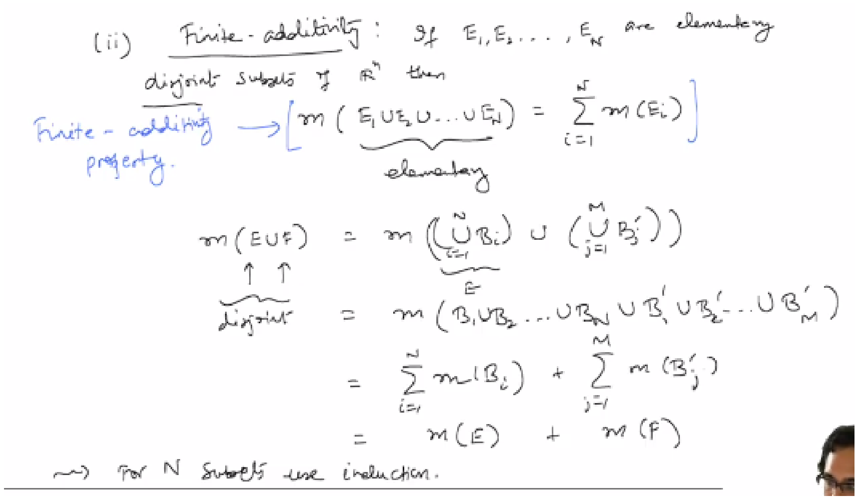
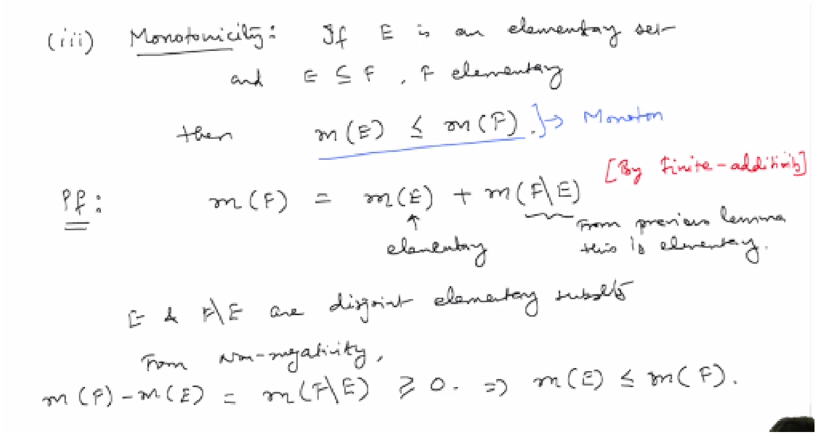
And for any other elementary set it can be either 0 or only a positive number because we have defined our measure of the intervals I_i to be a positive number and always finite. We are only considering bounded intervals, so therefore our length of those intervals will always be finite. So, we have a non-negative finite number for our elementary measures $m(E)$.



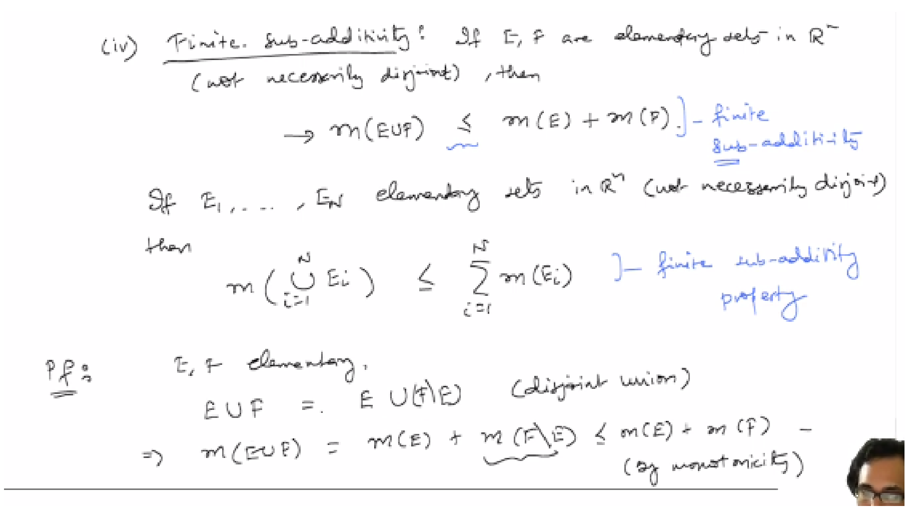
(Refer Slide Time: 03:40) The second property is finite additivity which we saw in the earlier next lectures that this is something geometrically reasonable to expect and it is what we aspire to have and the way we have defined it, of course it conforms to our geometric notion of length or volume or area, but now we will see that it will also satisfy our finite additivity property. If $E_1, E_2, \ldots, E_n$ are elementary disjoint subsets of $\mathbf{R}^n$, then

$$m(E_1 \cup E_2 \cup E_N) = \sum_{i=1}^{N} m(E_i)$$

We have seen this property for two sets E and F, but of course by induction you can prove it: union of finitely many elementary sets is elementary and the elementary measure of the union is simply the sum of the elementary measures of each individual E_i's, when you have disjoint elementary subsets. This is known as finite additivity. Let us say E is the union of B_i's and F is a union B_j'. Then $E \cup F$ is union of B_i's and B_j''s. Now this is a finite union of disjoint boxes and so we have seen that this measure of the union is the sum of measure of the individual boxes. Of course, for two subsets, it is easy to show and for N subsets we use an induction argument to show the same formula holds. This is the finite additivity property of elementary measures.



(Refer Slide Time: 08:01) Now from the first two property, we can deduce another property which is monotonicity. This says that if E is an elementary set and E is a subset of an elementary set, F is also an elementary set, then the elementary measure of E is less than or equal to the elementary measure of F. Let us see how we can deduce it from the first two properties. First of all, the measure of F can be written as the measure of E union the measure of $F - E$, note that here E is elementary. We have seen from previous lemma that $F - E$ is also elementary set. So, the elementary measure is again well define. Now E and F−E are disjoint elementary sets. Therefore, we have the measure of F minus the measure of of E is non negative. Hence we get measure E is less than or equal to measure F. Therefore, our elementary measure satisfies the monotonicity property.

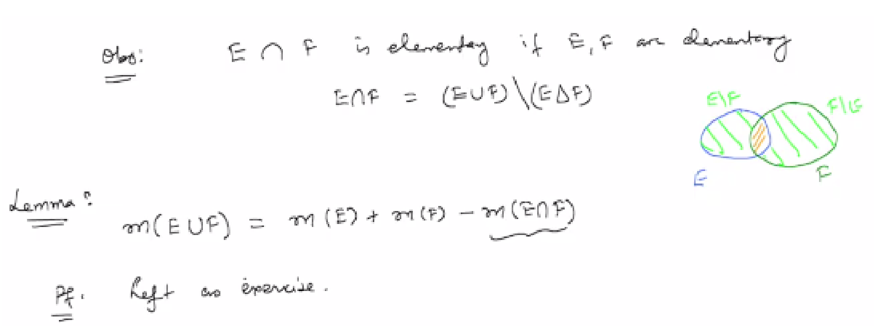


(Refer Slide Time: 10:51)Now, the next property is called finite sub-additivity. We have seen what is finite additivity, but this is finite sub-additivity by which we mean the following. If E, F are elementary sets in $\mathbf{R}^n$, not necessarily disjoint this time. Then their union $E \cup F = E \cup (F - E)$ is again elementary. Then $m(E \cup F) \leq m(E) + m(F)$. This is called the finite sub-additivity property. So, the difference here is that there is a less than or equal to sign rather than equal to sign, which appears in the finite additivity property. Because it is less than or equal to, it is called finite sub-additivity. Now, if you do an induction on the number of elementary sets, so if you have $E_1, \ldots, E_n$ elementary sets in $\mathbf{R}^n$, again not necessarily disjoint, then the measure of the union of all these sets is less than or equal to the sum of measures of their individuals. This is a generalization to any finite number of elementary sets and this finite sub-additivity property still holds. So, let us see the easy proof. So, here let me just prove it for two elementary sets E and F. Now, let me write $E \cup F$ as $E \cup (F–E)$. Now, these two are disjoint, so this is a disjoint union of elementary sets. Then

$$m(E \cup F) = m(E) + m(F–E)$$

and hence $m(E \cup F) \leq m(E) + m(F)$, because $F - E$ is a subset of F. So, by monotonicity we get this immediately.

One could also use other different decompositions using the intersection.



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Observation: If you have E and F to be elementary, then $E \cap F$ is also elementary. This is because $E \cap F$ is $(E \cup F) \setminus (E \Delta F)$. Now we have the following lemma whose proof is left as an exercise.

Lemma: $m(E \cup F) = m(E) + m(F) - m(E \cap F)$.

As a consequence of this lemma one can see the finite sub-additivity property because $m(E \cap F)$ is non negative.

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(v) Translation-invariance: for E elementary, $x \in \mathbb{R}^n$

$m(E+x) = m(E)$.

(Exercise).

Now, for the last property, we have already seen that the elementary measurement on boxes is translation invariant, but it will also of course work for any elementary set. So, for elementary set E and $x \in \mathbf{R}^n$, we have $m(E + x) = m(E)$. So, this is very easy to prove and I also leave this as an exercise. We stop our lecture here, and in the next lectures, we will introduce the concept of Jordan measure and Jordan measurability.