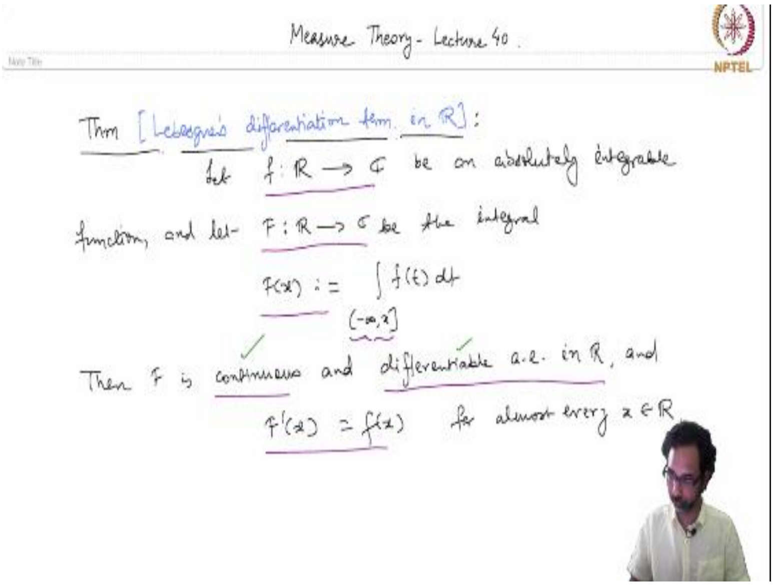
Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science – Madras

Lecture – 69
Lebesgue's Differentiation Theorem Statement and Proof - Part 1

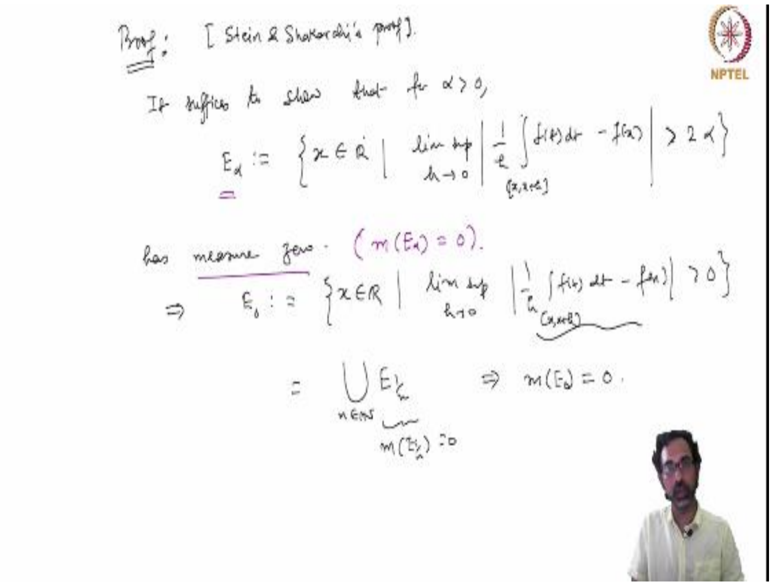
(Refer Slide Time: 00:16)



In this lecture we will look at a proof for Lebesgue differentiation theorem in R which was stated in the last lecture. So it says let me recall it says that if you take an absolutely integrable function on R and if you define its associated integral function capital F which is the integral over the interval – infinity to x of f t d t then this integral function is first continuous and then differentiable almost everywhere in R and the derivative is equal to the original function f x almost everywhere in R.

So we have already shown that this is continuous in the last lecture and we will continue our proof and take care of differentiability almost everywhere in this lecture which requires substantially more work than continuity.

(Refer Slide Time: 01:20)

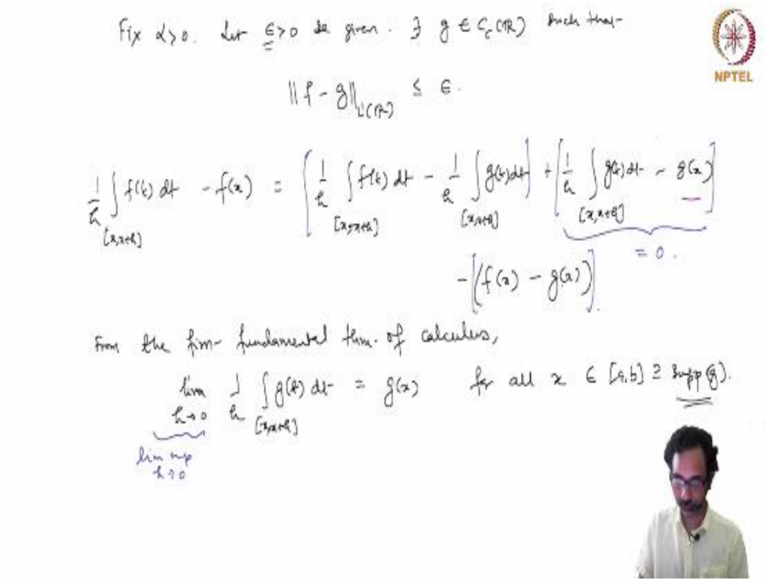


So let us look at the proof. So, I will follow the proof outlined in Stein and Shakarchi. So I will follow their method to prove this result. So, the first step is to note that it suffices to show that for any alpha > 0 the set E alpha defined as follows. So this is the set of all x in R such that the lim sup as h goes to 0 1 over h integral x to x + h f t d t – f x and you take the modulus of this is greater than 2 alpha has measure 0.

Because this would imply that E 0 which is the set of all R such that the set of all x in R such that the lim sup h goes to 0 and the same expression as before x x + h f t d t – f x is greater than 0 is nothing, but the countable union of E 1 over n in N and since each has measure 0 then it would follow that m E 0 is 0 which means that this two things are equal as limit h tends to 0 almost everywhere in x.

So it suffices to show this for this set E alpha that it has measure 0. So m E alpha = 0. So, let us try to see how we can show that m E alpha = 0 for any alpha > 0.

(Refer Slide Time: 04:20)



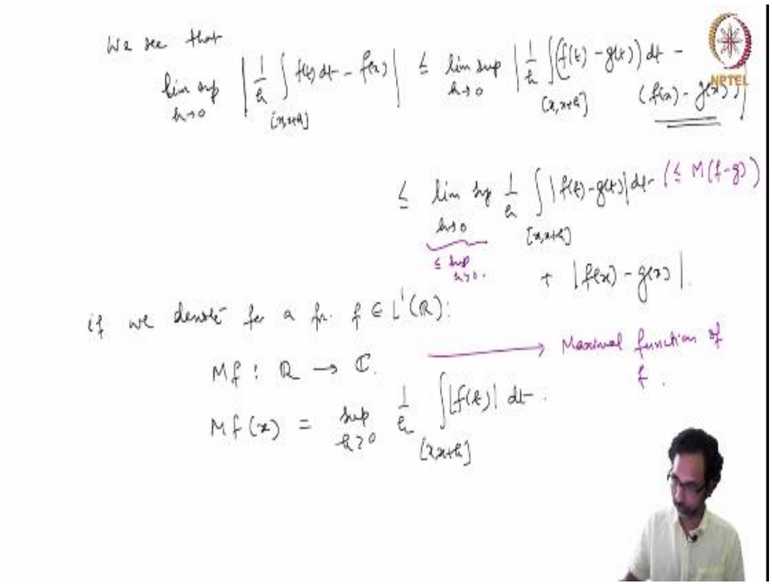
So, for this let us fix alpha for the rest of this proof and let epsilon > 0 be a given arbitrary number. Now we know that the compactly supported continuous functions are dense in L 1 functions. So there exist g in c c R such that so far this given epsilon there exist g such that the L 1 norm of f – g L 1 R is less than or equal to epsilon. So this we have already seen that the continuous compactly supported functions are dense in the L 1 norm and so we can choose this g for given epsilon.

And now if we write integral 1 over h integral x, x + h f t d t – f x. Now I am going to add and subtract some relevant terms f t d t – 1 over h x to x + h g t d t then I have to add again 1 over h x to x + h g t d t and then again I am going to subtract g x so – g x – f x and then again + g x so I am putting it in the bracket with the f – f so it becomes minus. So I am plugging in this two terms 1 over h integral g t d t over x to x + h and g x.

So, now we know from the first fundamental theorem of calculus that we saw in the last lecture that the limit as h tends to 0 1 over h integral x to x + h g t d t is equal to g x for all x which is (()) (07:26). So, I can take the support of g as the compact set on which x can move or I can take some interval a b which contains the support of g. So this is a compact set so you can always have a finite (()) (07:55) interval a b which contains this support of g.

So on this set we have this that the limit as h tends to 0 1 over h integral g t over x to x + h is equal to g x. So the term in the middle if you take the lim sup on both sides the term in the middle is going to vanish because this is the same as taking lim sup as h tends to 0 so this is going to give you 0 this whole thing is 0. So we are left with these two terms so we will deal with them separately.

(Refer Slide Time: 08:54)

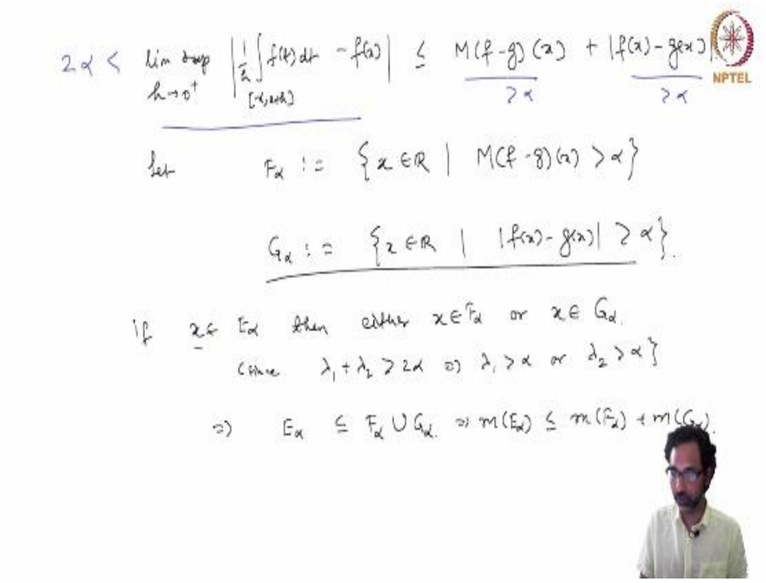


So we see that lim sup as h tends to 0 modulus of 1 over h integral x to x + h f t d t – f x is equal to or rather is less than or equal to the lim sup as h tends to 0 1 over h integral x to x + h f t – g t d t – f x – g x and now another application of the triangle inequality will give you the lim sup h tends to 0 1 over h integral x to x + h modulus of f t – g t d t + so there is no h in this one so you simply get modulus of f x – g x.

So we get this inequality and now if we denote for a function f in L 1 of R if we denote M f this is again from R to C which is a function defined as follows. So this is the supremum as over h > 0 1 over h integral x to x + h f t mod d t. So this function has a name this is called the maximal function of f and it is something like a supremum over all the average values of that f takes over intervals containing x on the right hand side so only right handed intervals containing x.

So if we have this then we can bound the first term which is lim sup as h tends to 0 by the supremum as of h > 0. So this term here is bounded above by the maximal function of f – g because you are taking the lim sup as h goes to 0 and this is less than or equal to the supremum over all h greater than 0 of the same expression. So for this maximal function.

(Refer Slide Time: 12:41)



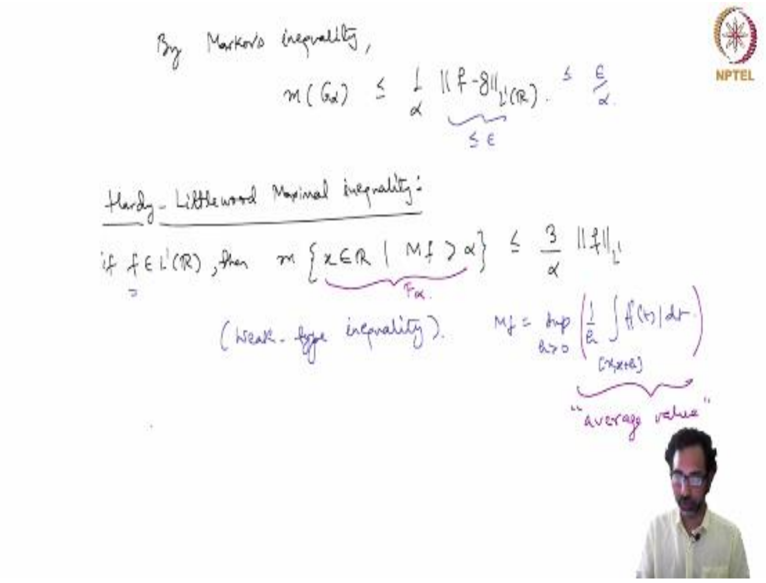
Let us write first what we get. So we get lim sup h goes to 0 so again all this h goes to 0 at the time taking should be from the right hand side. We can also use the left hand side, but let me just take the right handed limits everywhere I am going to take the right handed limits. So, with this maximal function so I get this inequality 1 over h f t d t over x, x + h – f x modulus is less than or equal to M f – g at x + modulus of f x – g x.

So now if we let F alpha to be the set of all x in R such that the maximal function of f – g x is greater than alpha and g alpha is the set of all x in R such that mod of f x – g x is greater than alpha. So, now we note that if x belongs to E alpha then either x belongs to F alpha or x belongs to G alpha because note that for F alpha we have this inequality that this left hand side given by the lim sup is greater than two alpha.

And so since if you have two numbers lambda 1 and lambda 2 and if you add them and if you get greater than 2 alpha this implies either lambda 1 is greater than alpha or lambda 2 is greater than alpha. So either the first term is greater than alpha or the second term is greater than alpha which means that if x is E alpha then x belongs to either f alpha or g alpha which is another way of saying that.

E alpha is contained in the union of F alpha and G alpha which means that the measure of E alpha is less than or equal to the measure of F alpha + the measure of G alpha. Now the measure of G alpha is quite easy to estimate by Markov's inequality.

(Refer Slide Time: 15:47)



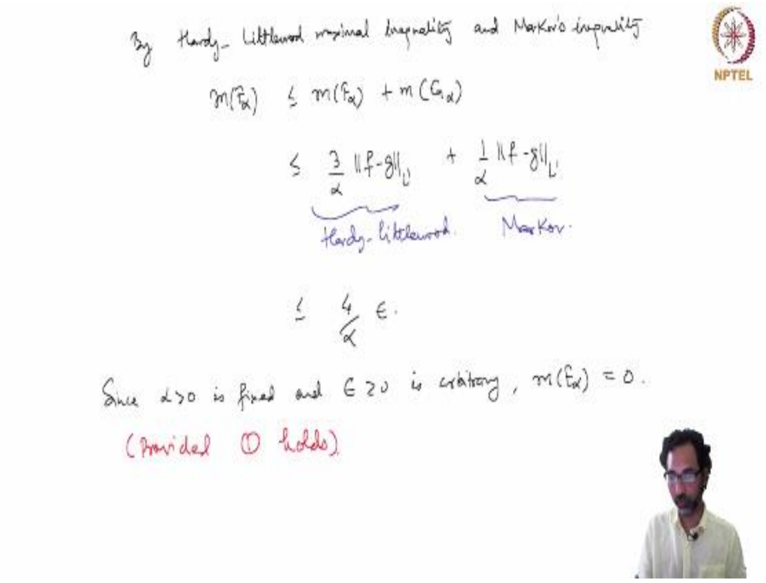
So by Markov's inequality we get the measure of G alpha is less than or equal to 1 over alpha L 1 norm of f – g and so this term is actually so for the second term in the last inequality here. So this term is less than or equal to epsilon / alpha because this is less than or equal to epsilon g was chosen to be less than such that L 1 norm of f – g is less than or equal to epsilon. So, the measure of g alpha is less than or equal to epsilon / alpha.

Now the main thing to show is that the measure of F alpha satisfies the similar inequality and this inequality has a name this is called the Hardy-Littlewood Maximal Inequality and it says that the maximal function so if f is in L 1 of R then the maximal function so rather I should say that the measure of the set of points in R such that the maximal function of f > alpha is less than or equal to 3 over alpha L 1 norm of f.

So this is a kind of inequality these are called weak type inequalities and these are extremely useful in doing analysis when you are trying to measure the variation of this L 1 function f when you vary the integral over large intervals well the average of the integral values. So because M f is the supremum over h > 0 1 over h x to x + h f t mod d t. So, this is this term that you are taking the supremum of is some kind of average value that the function f takes over large intervals.

So this Hardy-Littlewood Maximal Inequality is giving us the required estimate because this is nothing, but F alpha. So, let us suppose that the Hardy-Littlewood Maximal Inequality holds and then let us finish the proof.

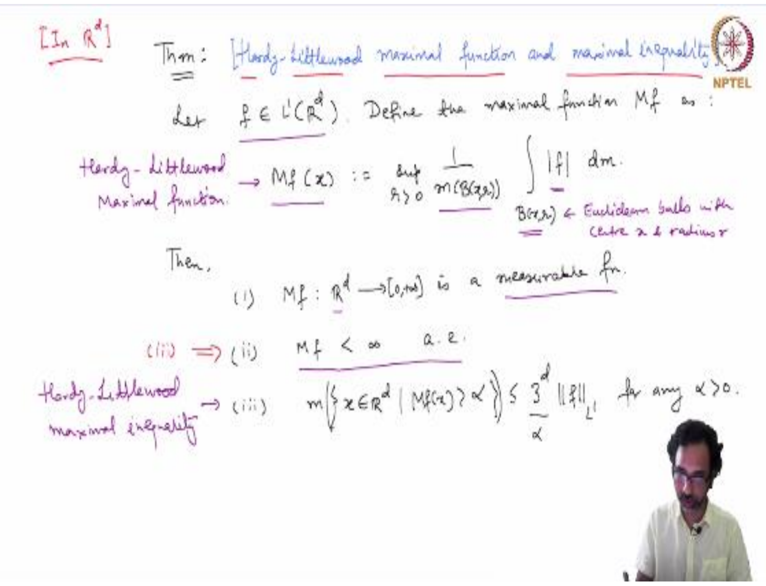
(Refer Slide Time: 19:27)



So by Hardy-Littlewood Maximal Inequality and Markov's inequality we get that the measure of E alpha which is less than or equal to measure of F alpha + measure of G alpha this is less than or equal to 3 over alpha norm f – g L 1 norm + 1 over alpha norm of f – L 1 norm of f – g. So this first term is by Hardy-Littlewood and the second term is by Markov. So, then we get this is equal to less than or equal to 4 over alpha times epsilon.

And so since alpha is fixed and epsilon is arbitrary we get the measure of E alpha equals 0 and this shows the statement for Lebesgue differentiation theorem. So now we still have to show that this inequity holds. So this inequality 1 holds so provided 1 holds. So, let us go into the detail of the proof of the Hardy-Littlewood Maximal Inequality.

(Refer Slide Time: 21:24)



So, let us see a generalization of our needed result where we can state this for general R d rather than just the one dimensional case and this is again the Hardy-Littlewood Maximal function and Hardy-Littlewood Maximal Inequality. So, if you take an L 1 function in R d then you can define the maximal function in a very similar way to what we did earlier. So we are taking in first the integral of mod f over Euclidean ball so these are Euclidean balls with center x and radius r.

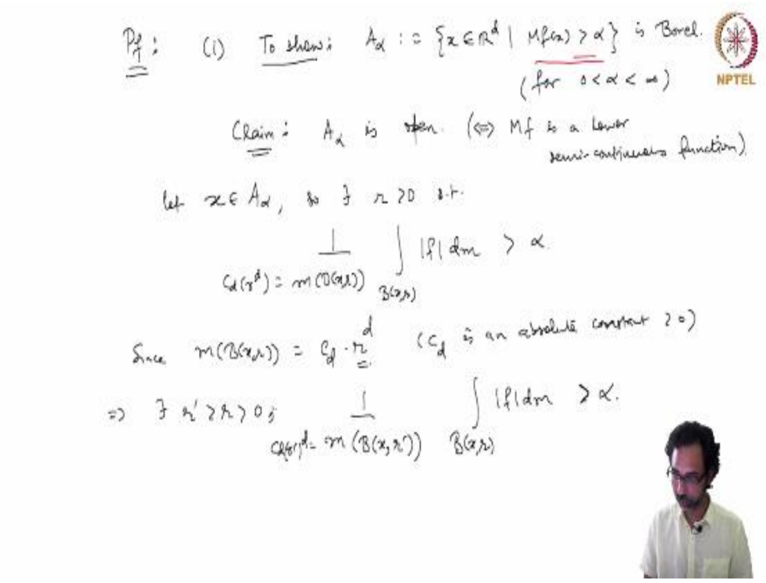
So take the integral over such balls and then you divide it by the measure of this Euclidean ball and then you take the supremum over all positive radius r > 0 what you get is called the Hardy-Littlewood Maximal function so this is the Hardy-Littlewood Maximal function and so the statement of this theorem says three things. First is that the Hardy-Littlewood Maximal function which is defined as a function from R d with values in c this is a measurable function.

Second is that it is finite almost everywhere in R d and the third is the Hardy-Littlewood Maximal Inequality so this part is the Hardy-Littlewood Maximal Inequality which says that the measure of points in R d for which M f there should be an x here M f x > alpha is less than or equal to 3 to the power d over alpha times the L 1 norm of f. So this is the main third part is the main part most significant part of this theorem.

And in fact 3 implies 2 so 3 we note it another color the 3 implies 2 because if you vary if you take alpha higher and higher and take the limit as alpha goes to infinity you get 0 on the right hand side and if you take the union on the left hand side you will get the measure of the set of all x such that M f equals infinity. So in fact this maximal function actually takes values in 0 + infinity the extended non-negative (()) (24:50) because we are taking the absolute value inside the integral.

So then the second part makes sense that the maximal function is finite almost everywhere and if you take like I said the limit as alpha goes to infinity on the right hand side of the third part and the union of all these sets on the left hand side then you would get exactly part 2. So for part 1 let us do part 1 first so let us see a proof of this theorem.

(Refer Slide Time: 25:24)



So for the first part we have to show that the set A alpha defines as x in R d such that M f x is greater than alpha is a Borel set and in fact I make the following claim that so this is for any alpha positive and finite. So I claim that A alpha is open. So, in terms of functions this is equivalent to saying that M f is so called lower semi continuous function. So this is just a side remark and it is not important to go into detail of lower semi continuity and upper semi continuity, but let us see how to prove that A alpha is open.

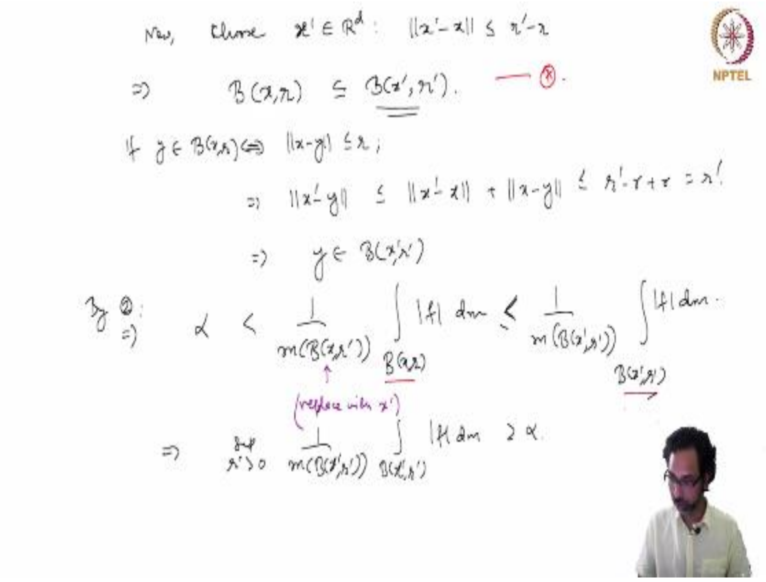
So let us take x in A alpha so there exist an r > 0 such that 1 over the measure of B x r integral over B x r mod f d m is greater than alpha. So this is by definition of the Hardy-Littlewood Maximal function which is the supremum over all such things. So, the maximal function is greater than alpha then there exist an r for which this is also greater than alpha by the properties of the supremum.

And since the measure of B x r is of the form alpha d times r to the power d so let me put c d times r to the power d in d dimensions the volume of Euclidean ball of radius r is proportional to r to the power d where c d is an absolute constant so it does not depend on x or r as to be a positive constant. So, this implies there exist an r prime greater than 0 such that 1 over m measure of B x r prime integral over B x r mod f d m is still greater than alpha.

So, I am just enlarging let me r prime > r I am just enlarging r so if you enlarge r so this is nothing, but c d times r prime to the power d and this is nothing, but c d times r to the power d. So, if you increase r the value on the left hand side of this inequality will drop because the value of the denominator has increased, but we can find r prime close enough to r such that the value does not drop too much and it does not go below or equal to alpha.

So, this is just by the continuity of this function 1 over r to the power d for strictly positive values of r. So we can find such an r prime such that this holds.

(Refer Slide Time: 29:29)



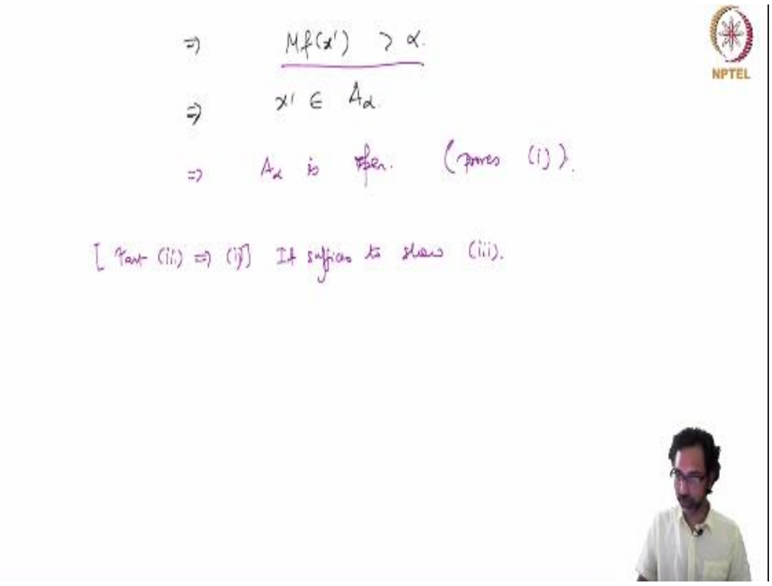
And now take choose x prime in R d such that the norm of x prime – x is less than or equal to r prime – r. So in that case we will have the ball centered at x with radius r is contained inside the Euclidean ball with center x prime and radius r prime because if y belongs to B x r which means that the norm of x – y is less than or equal to r and then this implies that the norm of x prime – y is less than or equal to x prime – x norm + norm of x – y.

So this is nothing, but r prime – r + r this is r prime which means that y belongs to the Euclidean ball centered at x prime and with radius r prime. So once we have this then we get this implies that so remember that we started out with this equation so this let me write this inequality as 1 so and let this be 2. So, by 2 we get alpha which is less than 1 over the measure of B x r prime integral B x r mod of f d m and this is less than 1 over measure of B x prime r prime.

So I can just replace here this x this x here replace with x prime because it was not changed the measure of the Euclidean ball because by translation in variance the measures of all this Euclidean balls is independent of x and secondly this is less than or equal to B x prime r prime mod f d m this is by this inclusion of B x r in B x prime r prime. So B x r is included in B x prime r prime so the integral is bounded above.

But on the right side what we get is this implies that the supremum over r prime > 0 1 over the measure of B x prime r prime integral over B x prime r prime mod f d m is greater than alpha because it is true for r prime. So it is true for all r prime > 0 because you are taking the supremum so if you take the supremum rather the supremum or all r prime > 0 this will still hold this inequality will still hold.

(Refer Slide Time: 33:25)



And so but on the left hand side we have M f x prime > alpha. So, this means that x prime belongs to A alpha and in fact I could have taken here an open ball around x with radius r prime – r and one would still have this chain of inequalities here we would have strictly less than or equal to r, but here may be strictly less than r, but this still holds and so finally we will get that.

We have found a Euclidean ball of radius r prime – prime around x for which any point lying in the Euclidean ball will satisfy M f x prime > alpha. So this means that A alpha is open. So this proves the first part and as we noted above that part 3 implies part 2 so it is suffices to show only part 3 now and so this is the most significant part of the proof and it requires another result which is called Vitali covering lemma.