Measure Theory
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Lecture – 68
Lebesgue's Differentiation Theorem Introduction and Motivation

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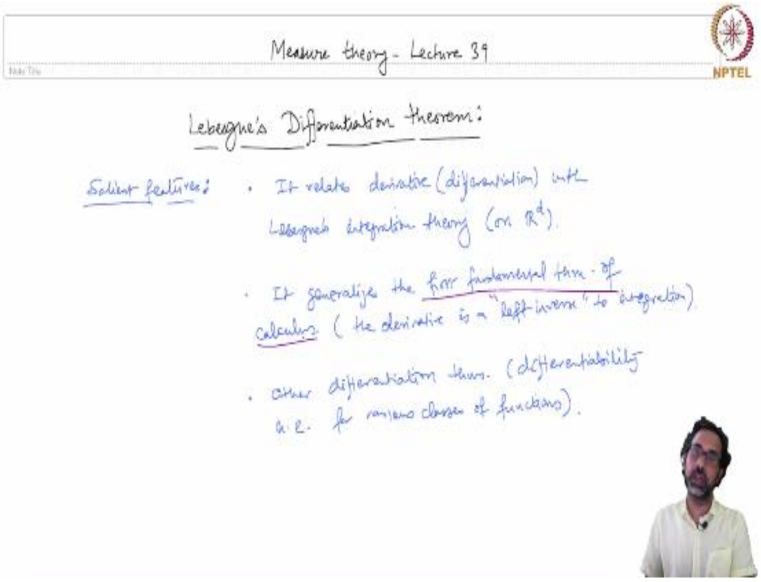

So now we come to a new topic which is Lebesgue differentiation theorem. As the name suggests this differentiation theorem relates the concept of derivatives with Lebesgue integration theory and now restrict ourselves to the case of the Euclidean space R d and then we will see that from calculus we know that derivative is inverse process of the integral and vice-versa.

So Lebesgue differentiation theorem generalizes this so called first fundamental theorem of calculus for absolutely integrable functions. So we put the salient features of Lebesgue differentiation theorem. So, first is that it relates the derivative or differentiation with Lebesgue integration theory. So on R d now we are back to Euclidean space and secondly it generalizes the first fundamental theorem of calculus.

So, we have learned this in Riemann's integration theory that the derivative of an integral is again the original function. So, roughly that derivative is a left inverse to integration. So, we will make this things precise and then it also deals with a number of other differentiation theorems. So this differentiation theorem gives criteria for functions to be differentiable outside of null set.

So, differentiability almost everywhere for various classes or functions. So, of course these (()) (03:35) with three salient features make up quite large number of results actually and to go into the depth of each result would not be possible due to lack of time and in fact we will be only be concerned with this topic for the rest of our course and this will be last course for this topic as well.

So, let us see how Lebesgue differentiation theorem generalizes the first fundamental theorem of calculus. So let me recall what is the notion of differentiability and what is the first fundamental theorem of (()) (04:07) differential function.

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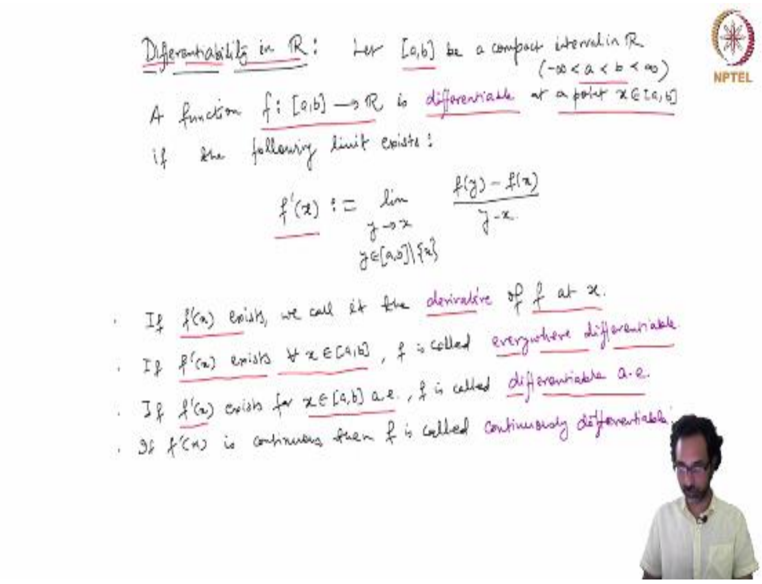

So let us look at the definition of differentiability of a real valued function define on a compact interval in R. So this is differentiability in R differentiability in R so we picks a compact interval a, b so here both a and b are finite numbers real numbers and we consider a function f from a, b through the real line so this function is called differentiable at a point x in a, b if the following limit exists.

So this limit is denoted by f prime x if it exists and this limit is given by the f y – f x over y – x and y approaches x where y is through any point any sequence of points in the interval a b minus this point x. So when this limit exists if f prime x exists we call it the derivative of f at x and if f prime x exists for all x in this interval a b it is called everywhere differentiable and finally if f prime x exists for x in a b almost everywhere.

So outside of a null set in a b with respect to the Lebesgue measure then f is called differentiable almost everywhere and finally one more terminology is that if f prime x is continuous then f is called continuously differentiable. So, again either everywhere or almost everywhere depending on the context. So, we have these terminologies and now we are ready to state the first fundamental theorem of calculus.

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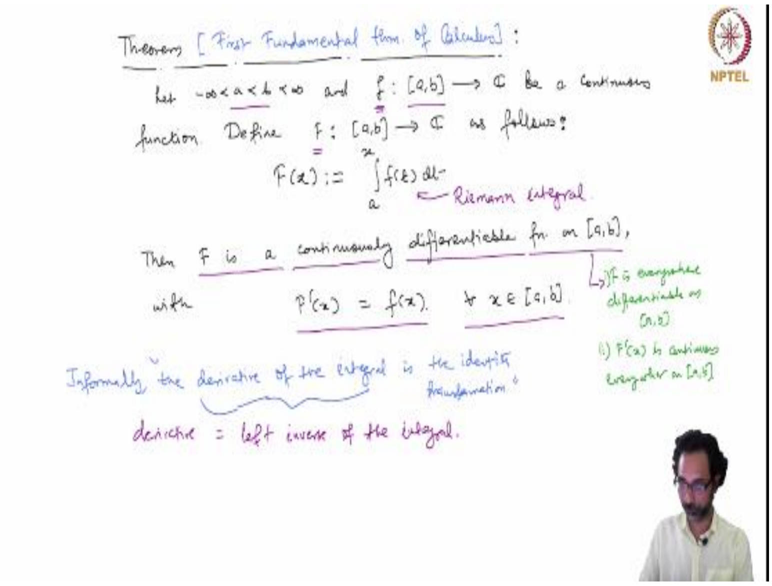

So now let us look at the statement for the first fundamental theorem of calculus and it says that if you take again a finite interval in R a b and you take a continuous complex valued function f and define capital F so we will first take small f to be continuous function and define capital F as the integral of this function f from a to x. So, this here is a Riemann integral because it is a continuous function so on a compact interval of the Riemann integral we have defined.

And so for any x in a, b you can define this function which is the Riemann integral coefficient for small f t and now the statement for the first fundamental theorem says that now if you differentiate capital F then you should get back (()) (07:45) small f. In fact capital F is a continuous differentiable function on the entire interval a b so it is first it is everywhere differentiable so there is two parts.

So first is that f is everywhere differentiable on a b so this is the first one and the second one is that F prime x is continuous everywhere on a b it is continuous set all points of a b. So this is what we mean when we say that capital F is a continuously differentiable function on a b and with derivative F prime x. First of all it is a differentiable function so the derivative exist everywhere and derivative is precisely the function small f that we began with here and this is true for all x and a b.

So this is the first fundamental theorem of calculus and this is exactly what I mentioned before that the derivative. So, informally so the derivative of the integral is the identity transformation on functions. So, I will just putting it in quotes so if you remove these integrals and derivatives as operator which takes a function and gives you back another function then the derivative is the left inverse.

So this implies that the derivative is equal to the left inverse of the integral. So this is what we mean when we say the integral is an anti-derivative which is precisely in the sense that it is the left inverse of this integral. Now there also exist a second fundamental theorem of a calculus which gives you sufficient criteria for the integral to be a left inverse for the derivative or the derivative to be a right inverse for the integral which we will come to later.

So, this is the first fundamental theorem of calculus and so let us look at a quick proof for this theorem.

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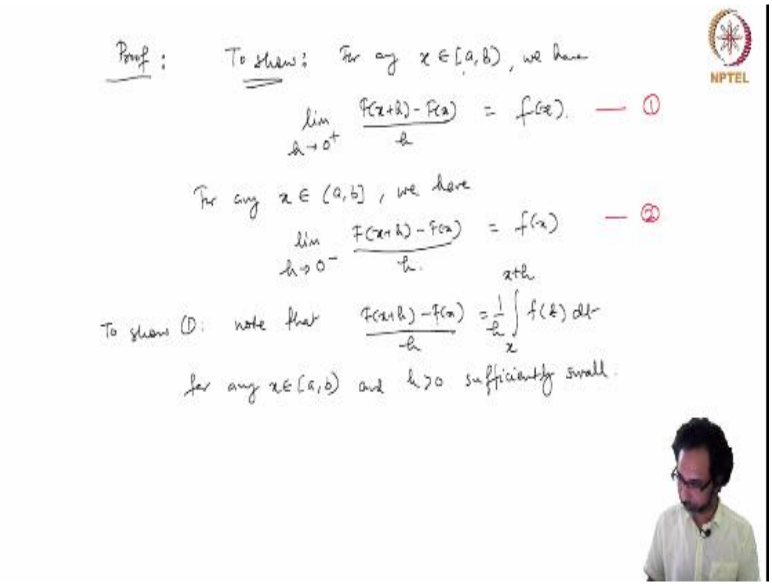

So, let us look at a proof of this theorem. So we have to show that for any x in a b with a included in b excluded we have that limit as h tends to 0 + F x + h – F x over h is equal to f x. Noting that here we can only approach a from the right so it is a right handed limit with h approaching 0 from the right. Similarly for any x in a b with a excluded and b included we should have then the limit as h tends to 0 – so from the negative sign approaching 0 from the negative sign F x + h – F x over h is equal to f x.

So, we have to deal with these two situations differently because we have end points a and b and one can only approach a from the right and b from the left. So, let us denote the first equation as 1 and the second equation as 2. So, I will just prove 1 and the proof of 2 is similar. So to show 1 we note that F x + h – F x over h is equal to the integral from x to x + h f t d t over h.

So this is for any h positive sufficiently small and x in a b rather I should first write for any x in a b and h positive sufficiently small because h the magnitude of h will depend on x because we need to have this integral define for values of x between a and b. So, h should be sufficiently small depending on the value of x so that x + h is not greater than b.

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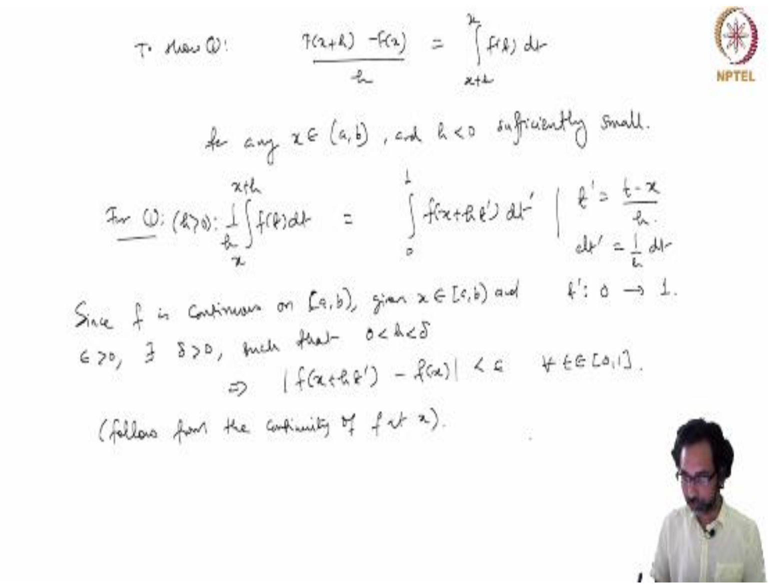

Similarly to show 2 we can say that F x + h – F x over h is equal to the same thing, but we would rather have x + h to x f t d t for h less than 0. Again for any x in a b with a excluded and b included and h less than 0 sufficiently small. So, now we can rewrite the first one so for one we can write the integral so remember that here h is positive so integral from x to x + h f t d t can be written as integral 0 to 1 f x + h t prime d t prime and even you can have a 1 over h here. So this is by a change of variables t prime is t – x over h.

So, first of all d t prime equals to 1 over h d t and t prime goes from 0 to 1 because t goes from x to h. So here we have a change of variables and a new integral from 0 to 1 and with the function f x + h t prime d t prime. So we are going to use the continuity of f. So, since f is continuous on a b given x in a b and epsilon > 0 there exists a delta positive such that 0 < h < delta implies that f x + h t prime – f x is less than epsilon for all t in 0, 1. So this follows from the continuity of f at x.

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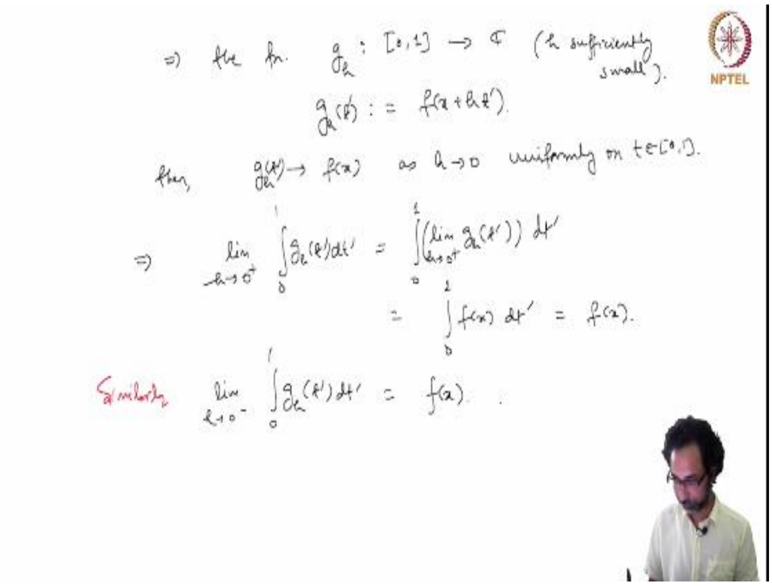

So this means that the function if we define a function g h for h positive and sufficiently small from 0, 1 to c given by g h t as f of x + h t or rather t prime let us say then g h converges g h t prime converges to f x as h tends to 0 uniformly on t in 0, 1 because if we go back to this then this delta that is chosen does not depend on t and only depends on x. So (()) (18:22) is uniform over t and 0, 1.

And so this implies that the limit as h tends to 0 + integral 0 to 1 g h t prime d t prime is equal to the integral 0 to 1 limit h tends to 0 + passing the limit inside the integral of d t prime because it is a uniform convergence. So for Riemann integration you can interchange the order of limit and integration and this is nothing, but integral 0 to 1 f x d t prime and this is just f x.

And similarly we can prove so this is remember that g h was defined for h sufficiently small and fixed and x in a b. So here we have already fixed an x in a b and then proven our delta and our h so that we have uniform convergence of this function g h t prime as h tends to 0 they all converges to f x. So we have this for the positive side and similarly one can show that limit h tends to 0 – 0 to 1 g h t prime d t prime = f of x.

So this shows that the function capital F is continuous and that derivative of the continuous function capital F is simply small f of x.

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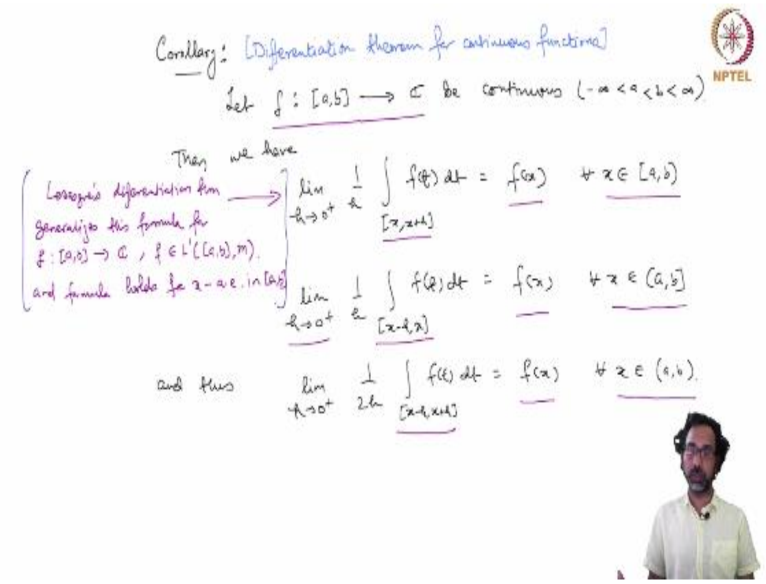

So from the proof we obtain this following corollary which is called differentiation theorem for continuous functions and it says that f is a continuous function on a compact interval with values in the complex numbers then we have that the limit of h going to 0 from the right side 1 over h the integral of f over the interval x to x + h is exactly f of x this is for all x in a b where a is included and b is excluded.

Similarly, the limit as h tends to 0 + 1 over h (()) (21:25) x – h to x f t d t is again f x and now this is for all x in a b where b is included and a is excluded and if you combine these two we get the limit as h tends to 0 from the right 1 over 2 h and integral over x – h to x + h of f t d t is again f x this is for x in the interior for the interval a b. So this is the differentiation theorem for continuous function that one learns in classical calculus.

And the Lebesgue differentiation theorem and we are seeing as a generalization of this formula for example. So, Lebesgue differentiation theorem generalizes this formula for f when we no longer have a continuous function, but f is in L 1 of a b with the Lebesgue measure. So, it is an absolutely integrable function on this compact interval a b and this formula holds for x almost everywhere in a b.

So this holds only outside a null set in a b. So we will see a proof of differentiation theorem in dimension 1 which is for the (()) (23:29) and then we will see also its generalization to high dimensions, but for both these we will need the so called Hardy–Littlewood Maximal inequality which gives us the required technology to prove this Lebesgue differentiation theorem which you will see now.

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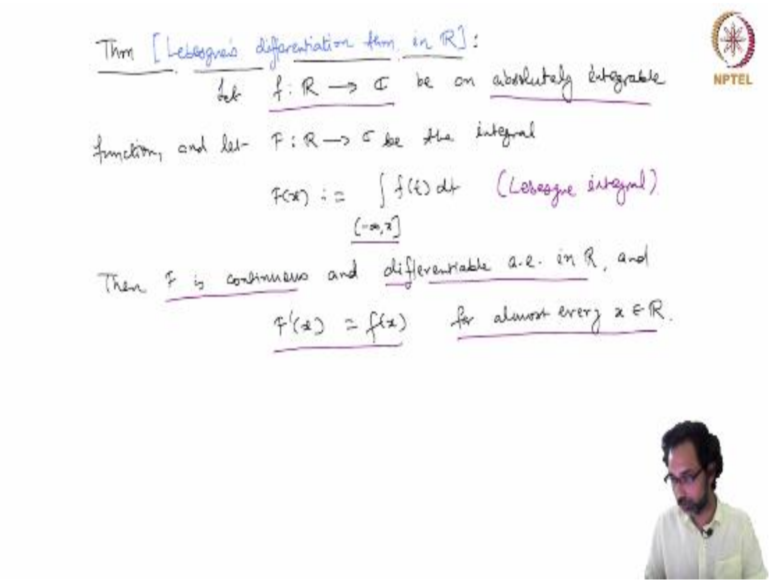

So, let us see there are statement of Lebesgue differentiation theorem in R. So it says that if you have f from R to c now we are no longer confined to a compact interval as before because we are in Lebesgue integral setting. So, we are free to choose unbounded domains and so we choose a function complex valued measurable function which is absolutely integrable over R and now we define this capital F function has been put which is the integral of f t over the interval – (()) (24:37).

And this is now a Lebesgue integral and now the statement the assertion of the theorem is that f is continuous this is the first part and secondly it is differentiable almost everywhere in R so it is differentiable for x outside of null set in R and finally probably most importantly the derivative of capital F and x is equal to f of x for almost every x in R assuming this holds for x outside of a null set in R.

So note that we have relaxed our assumption that this be continuous and now we only consider absolutely integrable functions, but now our assertion is also weaker than before because we are only asserting that this formula for F prime x capital F prime been equal to f x this holds for almost not everywhere in R. So simple example we will show that if you try to enforce F prime capital F prime x to be f x everywhere in R then this will easily fail.

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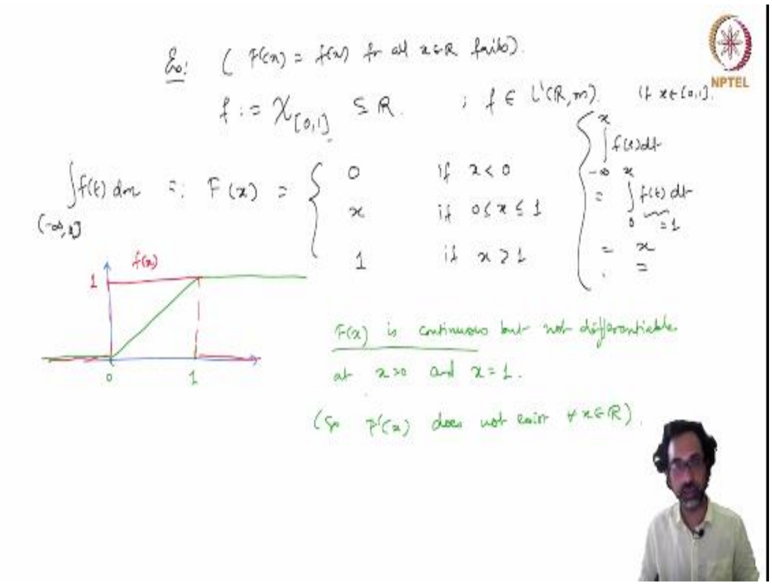

So let me give an example where so this is an example for which F prime x equal f x for all x in R fails. So, we can take f to be the function to indicate a function of the interval 0, 1 in R so this is again absolutely integrable function f belongs to L 1 of R and so now we can reduce what is capital F so capital F x is by definition the integral. So this is by definition the integral of f x f t d t over.

So I am writing d t but it is the same as d m so with the Lebesgue measure. So this is over the interval – infinity to x so if x is less than 0 then of course f is 0 because it is a indicator function and therefore this is going to give you 0. Now if 0 less than or equal to x less than or equal to 1 then one can easily compute that capital F of x is simply at x because this characteristics function of 0, 1 is integrable because 0, 1 is Jordan measurement.

So it is Riemann integrable and so the integral of – infinity to x f t d t so if x belongs to 0, 1 then this is equal to 0 to x f t d t and this is nothing, but x equals this is just the indicative function of 0, 1 so it is equal to 1 on this range 0 to x so this is simply x and now if x is greater than 1 then f x is nothing, but 1. So the graph of small f from 0 to 1 is simply this function which is 1, 0, 1 and 0 outside so this is small f of x.

But for capital F of x it is 0 outside for values of x between – infinity and 0 then it becomes x in the range 0 to 1 and then it becomes 1 outside. So we can see that F of x is continuous, but not differentiable and x = 0 and x = 1 and so we cannot speak of F prime x even does not exist for all x in R. So this example shows that we cannot hope to have the same result as before where F prime x exist for all x in a compact interval a b and it was equal to F x. So here this fails, but we still have the fact that F x is continuous so we can easily show.

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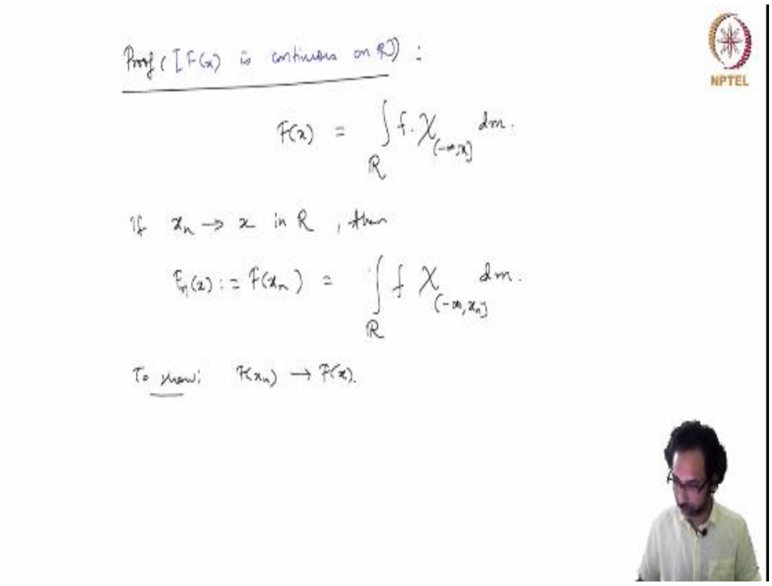

So this is the first part of the theorem proof of the fact that capital F is continuous on R. So, how do we prove this so this is easy because if you write F x as the integral of F times the indicative function of – infinity x d m then if you take any sequence x n converging to x in R then F x n. So let me write this as F n of x which is F x n and this is integral over R of the indicative function of – infinity to x n and so we have to show that as F x n converges to F of x and this is not very difficult because we can use the dominated convergence theorem.

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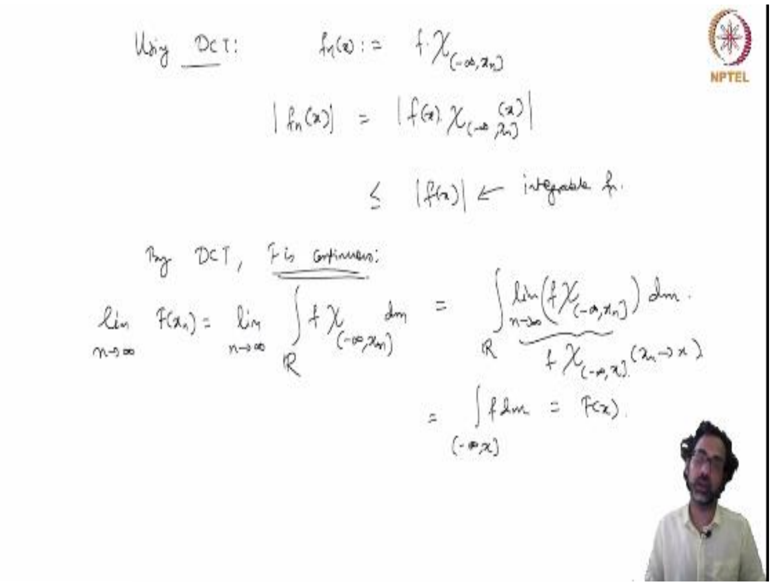

So using the dominated convergence theorem so for the sequence of functions f n x defined by f times indicative function of – infinity x n then we have mod of f n x which is mod of f x times indicative function of – infinity x n and this is less than or equal to mod of f x and this is integrable function. So this implies that by DCT we have F of x n which is so the limit as n goes to infinity F of x n which is the limit as n goes to infinity of the integral over R f chi – infinity x n d m and this is equal to the limit taking inside integral of the function that you get when integral limit in (()) (33:09) f of chi – infinity x n d m.

But this is nothing, but f of chi – infinity x because x n converges to x. So the indicative function of – infinity x n converges to – infinity x and so this is nothing but the integral – infinity x f d m and this is exactly F of x. So we see that F is continuous. So this proves the first part of the Lebesgue differentiation theorem of course the most difficult part is to show that f prime x equals f x almost everywhere and for that we need some more technical results what are called as Hardy–Littlewood Maximal inequality which we will see in the next lecture.