Measure Theory
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Module No # 13
Lecture No # 67
Fubini- Tonelli theorem: interchanging order of integration for measureable and L1 functions on sigma-finite measure spaces

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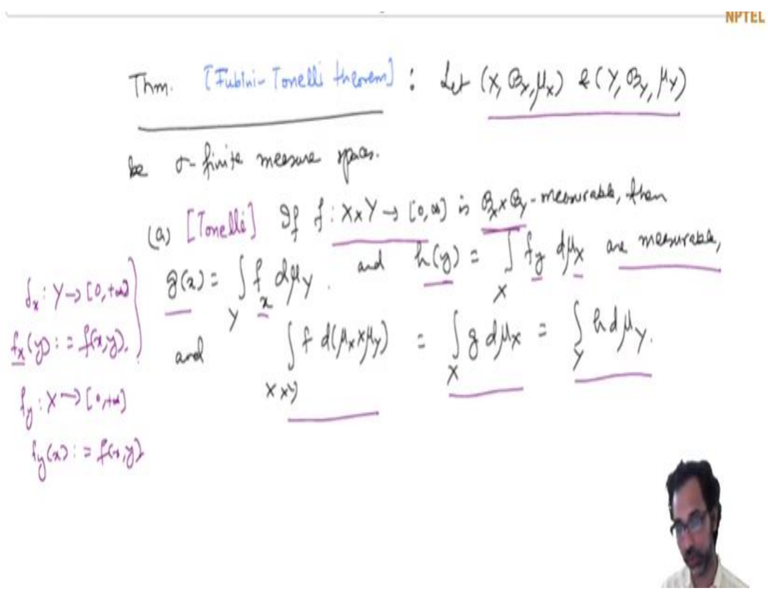

Now finally we come to the theorem’s of Fubini and Tonelli so again we fix sigma finite 2 measure spaces x and y which are both sigma finite. And so, the theorem of Tonelli’s is as follows so if you take a function f which is an unsigned measureable function with respect to the product sigma algebra B x cross B, y. Then the integral of the function f subscript x remember that fx was the function from y to the non-negative real’s in this case which was defined as f x y as f of x comma y.

So this was the definition similarly we can define f y to be given by f y evaluated at x is again f x y. So the integral of, f x the first one with respect to the measure d mu y is a function of x and similarly the integral of, f y with respect to d mu x is a function of y. And so these 2 functions g x and g y are measurable this is the first assertion in the theorem. And secondly if you integrate f with respect to the product measure over the product space x cross y this is the left hand side then this is equal to the integral of g with respect mu x and is equal to the integral of h with respect to mu y.

So this is Tonelli's theorem which is valid for measureable functions which are unsigned in the product measure in the product sigma algebra. So the theorem of Fubini is then for is a generalization of Tonelli's theorem when you have no longer unsigned but complex valued L1 functions in the product space.

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(6) [Fubini] If $f \in L^1(X\times Y, \mu_X\times\mu_Y)$ then $f_x \in L^1(Y,\mu_Y)$ for μ_X-a.e. $x \in X$, $f_y \in L^1(X,\mu_X)$ for μ_Y-a.e. $y \in Y$, and the a.e. defined functions

$$g(x) = \int_Y f_x \, d\mu_Y \quad \text{and} \quad h(y) = \int_X f_y \, d\mu_X$$

are in $L^1(X,\mu_X)$ and $L^1(Y,\mu_Y)$, respectively.

Moreover, we have that

$$\int_{X\times Y} f \, d(\mu_X\times\mu_Y) = \int_X g \, d\mu_X = \int_Y h \, d\mu_Y.$$

So this is the theorem of Fubini which says that if, f is an L1 function for the product measure then the function f subscript x is in L1 mu y, mu y. So it is integrable with respect to mu y absolutely integrable with respect to mu y for mu x almost everywhere x, so not for all values for x but only for x outside of a null subsets with respect to mu x. Similarly y belongs to L1 x mu x for mu y almost everywhere y in y.

And then the almost everywhere defined function g x and h y which are integrals of, f x and f y are in L1 of x and L1 of y respectively. So g belongs to L1 of x and h belongs to L1 of y and we again we have the formula that the integral with respect to the product measure is equal to the integral of g. And then the integral of h so this is a generalization of Tonelli's theorem for L1 functions on the product measure space.

So let us see the proof for Tonelli’s theorem which would imply the second part which is Fubini’s theorem.

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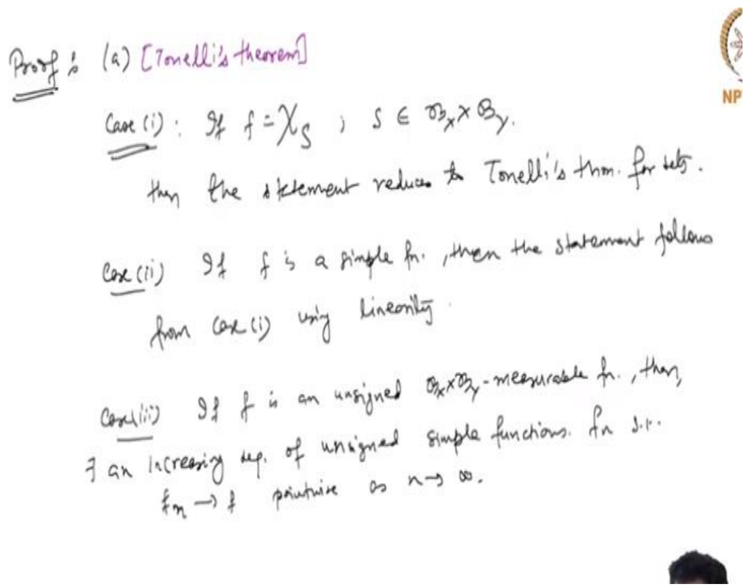

So for the proof of Tonelli’s theorem we can divide it into a few cases and it the first one will be simply the Tonelli’s theorem for sets with is the case when if, f is the indicative function of set S for some X belonging to B x cross B, y. So then Tonelli’s theorem the statement reduces to Tonelli’s theorem for sets this we have seen in the last lecture. So this takes care of the case when f is the indicative function of a measureable set.

Second case that if, f is a simple function again in the product space then the statement follows from the first case statement follows from the case 1. So again by Tonalli’s theorem for sets using linearity because you will have finitely many terms in the simple function. So this is finitely linear combination of characteristics sets characteristics functions of measureable sets. And so by linearity we get the result for simple functions.

And finally if, f is a general measureable function is an unsigned B x cross B, y measureable function. Then their exist the sequence of a increasing sequence of unsigned simple functions f n such that f n converges to f point wise as n tends to infinity. And so now since we have we already know this for simple functions then we can state the following results.

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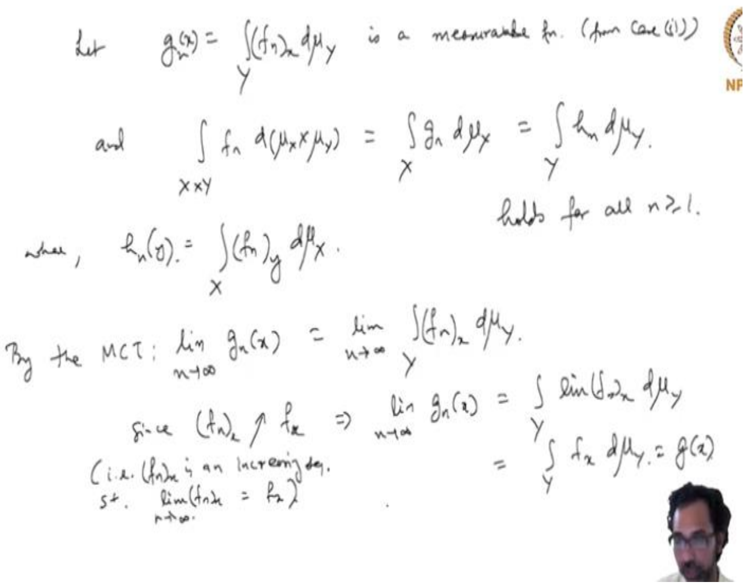

So let g n be the integral of f n x over x so g sorry this is the integral over y. So of course the statement says that this is measureable function. This is this follows from case 2 and we have that the integral of f n on the product space with the product measure mu x cross mu y it is equal to the integral of g n over d mu x. And this is also equal to h n similarly for the d mu y where h n is defined similarly this is integral over f n y d mu x so this is over y.

So this holds for all n greater than equal to 1 and now by the monotone convergence theorem we have that g n x the limit of g n x this is equal to the limit as n tends to infinity of these integral f and x d mu y. And since f n x increases point wise to f x and this means that this is an increasing sequence converging point wise to f x. So this is the notation I have used f n x is an increasing sequence of measureable functions.

Such that limit of, f n x equals f x as n goes to infinity. So by monotone convergence theorem we get that the limit of g n x as n tends to infinity this is equal limit as n tends to infinity g n x this is equal to the limit of, f n x the integral of the limit over y. And this is nothing but the integral of, f x so this implies that and so of course this is nothing but on the right hand side this is nothing but g of x.

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$\Rightarrow$ g is a measurable fn. on X.

Similarly, h is a measurable fn. on Y.

$$\int_X g \, d\mu_X = \int_X \lim_{n\to\infty} g_n \, d\mu_X \overset{MCT}{=} \lim_{n\to\infty} \int_X g_n \, d\mu_X$$

$$= \lim_{n\to\infty} \int_{X\times Y} f_n \, d(\mu_X \times \mu_Y)$$

$$\overset{MCT}{=} \int_{X\times Y} f \, d(\mu_X \times \mu_Y)$$

Similarly, $\int_Y h \, d\mu_Y = \int_{X\times Y} f \, d(\mu_X \times \mu_Y)$. (This proves Tonelli's thm.)

So this means that g x is measureable in x which means that g of x g rather is a measureable function on x. Similarly one can show that h is a measurable function on y and now if we integrate g with respect to x over the measure mu x then this is nothing but the limit as n tends to infinity g n d mu and this is again by monotone convergence theorem you can take limit outside. So this is limit n tends to infinity g n d mu x but this integral was the same as the integral of the product of, f n with respect the product measure.

And then again by the monotone convergence theorem we take the limit inside to get f d mu x cross mu y. So we see that the formula holds for g and similarly one can show it for h similarly h d mu y equals x cross y f d mu x d of mu x cross mu y. So this proves Tonelli's theorem in the general case.

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(b) If $f \in L^1(X\times Y, \mu_X\times\mu_Y)$ then $|f|$ is an integrable unsigned measurable fn. $\Rightarrow$ Tonelli's thm applies:

$$\|f\|_{L^1}^{X\times Y} := \int_{X\times Y} |f| \, d(\mu_X\times\mu_Y)$$

$$\overset{\text{Tonelli}}{=} \int_X g \, d\mu_X \quad \text{where} \quad g(x) = \int_Y |f|_x \, d\mu_Y = \int_Y |f_x| \, d\mu_Y = \|f_x\|_{L^1}^{Y}$$

if $\|f\|_{L^1}^{X\times Y} < \infty \Rightarrow \int_X g \, d\mu_X < \infty$

$\Rightarrow g(x) < \infty$ for μ_X-a.e. $x \in X$.

$\Rightarrow f_x \in L^1(Y,\mu_Y)$ for μ_X-a.e. $x \in X$.

So now for Fubini's theorem if f is in L1 of the product space then mod f is an unsigned measureable function is an integrable unsigned measurable function which means that Tonelli's theorem applies. And so we get that the L1 norm of, f over the product space x cross y this is equal to by definition integral of mod f over d mu over the product space. And by Tonelli's theorem we get that this is equal to the integral of x over x of g d mu x where g x is given by the integral of mod f x over y.

But this is nothing but the integral of the modulus of, f x d mu y and this is nothing but the L1 norm over y of the function f of x. So if this norm over the product space is finite this implies that the L1 norm first of all that this integral g d mu x is finite which means that g x is finite for mu x almost everywhere x in x. And this implies that this x is in L1 of x L1 of y mu y for mu x almost everywhere x in x. And similarly the result holds for f y so this shows that Fubini's theorem holds.

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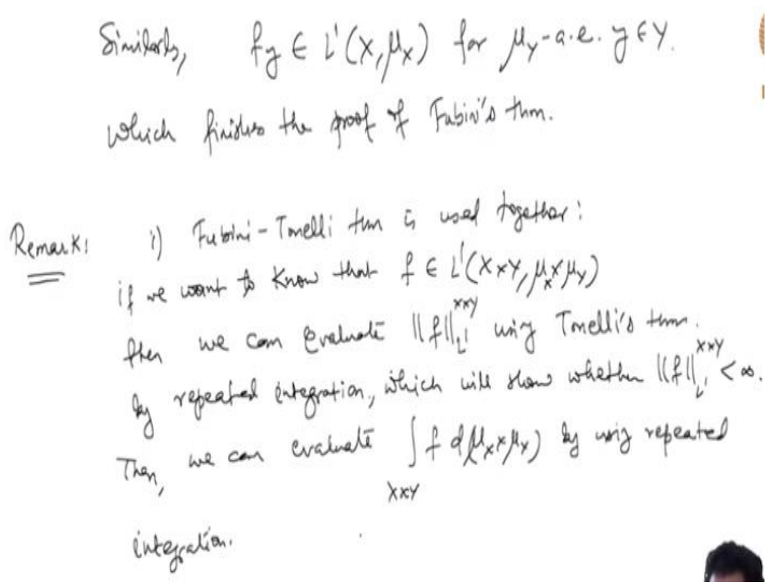

So similarly f y belongs to L1 over x over mu y almost everywhere y in y and which finishes the proof of Fubini's theorem. So now some remarks are in order so remark first is that the Fubini Tonelli's theorem is used together meaning that if you want to know. So if we want to know that f is in L1 of x cross y mu x cross mu y. Then we can evaluate the L1 norm over x cross y using Tonelli's theorem by repeated integration.

So this will show us which will show whether norm of, f L1 x cross y is finite or not because it is always easier to do repeated integration then direct integration over the product space. And then once you know that it is L1 then we can evaluate the integral over of, f over this product space by using repeated integration. So the change of order is then permitted and then we can use it.

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ii) If $X = Y = \mathbb{N}$, $f(n,m) = a_{n,m}$ (Double seq.).

then if $a_{n,m} \geq 0 \;\; \forall\, m,n \in \mathbb{N}$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{n,m} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sum_{(n,m)\in\mathbb{N}^2} a_{n,m} = \sup_{\substack{F \subseteq \mathbb{N}^2 \\ F \text{ finite}}} \sum_{(n,m)\in F} a_{n,m}.$$

For any arbitrary set X:

$$\sum_{x\in X} a(x) := \sup_{\substack{F \subseteq X \\ F \text{ finite}}} \underbrace{\sum_{x\in F} a(x)}_{\text{finite sum}}.$$

Another remark is that if you take x = y equal to natural numbers and f n m to be sum double sequence and a n m this is a double sequence. Then we have that if a, n, m is non-negative for all m, n then we can change the order of summation n = 1 to infinity a, n, m, m = 1 to infinity this is equal to 1 to infinity a, n, m. Then again m = 1 to infinity a, n, m and both are equal to that double sum in n square of a, n, m.

Here the definition for any arbitrary set x we can define the sum over so this a x is a function a, x made be a function of on x. And this equal to the supremum of over the finite subsets f is finite of x in f a, x. So this is a now a finite sum and then you the supremum so this is by definition and this is the definition we can use so this is nothing but supremum of over a finite subsets of n to finite.

Then we can use n, m in f a, n, m so now this is 2 finite sums then you can use the formula for finite sums and you can also interchange. So this holds for double summations of double infinite sequences.

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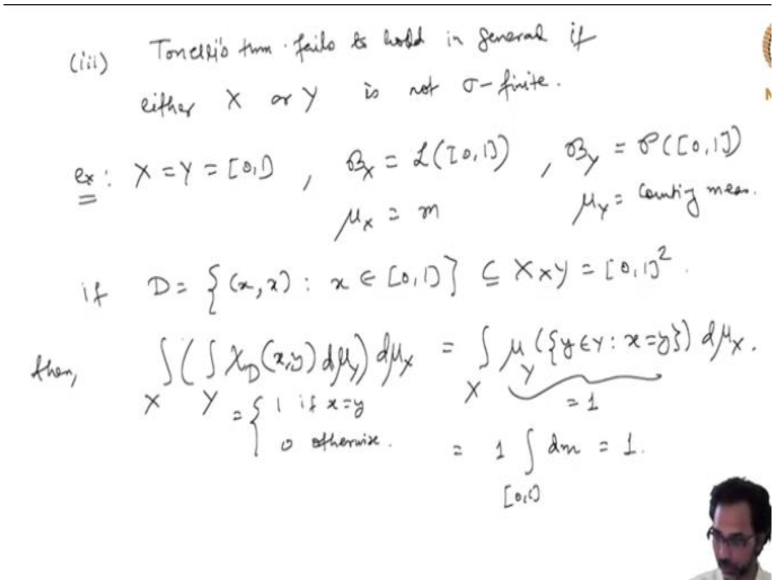

And then lastly Tonelli's theorem fails to hold in general if either x or y or is not sigma finite. So an example is x = y = 0, 1 and mu so B x can be taken to be the Lebesgue sigma algebra over 0, 1 and B, y and so mu x is the Lebesgue measure and B, y is the power set of 0, 1 with mu y the counting measure. And so if d is the diagonal in 0, 1 cross 0, 1 so this is x cross x, x such that x in 0, 1 this is a subset of x cross y which is nothing but 0, 1 square.

Then if we take the integral over x and then the integral over y of the indicative function of d so first we integrate with respect mu y which is the counting measure and then with respect to mu x which is the Lebesgue measure. Then this is equal to the following so note that Chi e x y this is equal to 1 if x = y and 0 otherwise. So this is the integral of the measure with respect to the counting measure of the set y in y such that x = y and d mu x.

And now this is just 1 because there is only one point which is equal to x over 0, 1 because x is fixed here. So the measure is equal to 1 so then you get just integral of Lebesgue measure over 0, 1 this is equal to 1.

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On the other hand,

$$\int_Y \left[\int_X \chi_D(x,y)\, d\mu_X \right] d\mu_Y = \int_Y 0 \cdot d\mu_Y = 0.$$

$$\Rightarrow \quad \underline{\int_X \int_Y \chi_D(x,y)\, d\mu_Y\, d\mu_X} \neq \int_Y \int_X \chi_D(x,y)\, d\mu_X\, d\mu_Y.$$

(Note that $Y = [0,1]$ with the counting measure μ_Y is not σ-finite)

On the other hand if you take the integral over y, first and then integral over x first of the indicative function first with respect to the Lebesgue and then with respect the counting measure. Then again for any fixed y this is going to be non- zero only when x = y so only 1 point in x contributes and this as Lebesgue measure 0 so therefore the inner one is 0. So this is simply 0 times d mu y and this is 0 so this means that these 2 things are not equal.

So Chi d x y d mu y d mu x is not equal to y x Chi d y Chi d x y d mu x d mu y and here note that the second space y with the counting measure mu y is not sigma finite. And so even though you this indicator function of d is, measureable still we do not have that the interchange of order of integration is allowed so in this case it fails to hold. So therefore this sigma finiteness is a necessary condition for this Tonelli's theorem to hold.

So let me finish the lecture here and in the next week lectures we will see a new topic which is the Lebesgue differentiation theorem.