Measure Theory
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Module No # 13
Lecture No # 66
Tonelli's theorem for sets – Part 2

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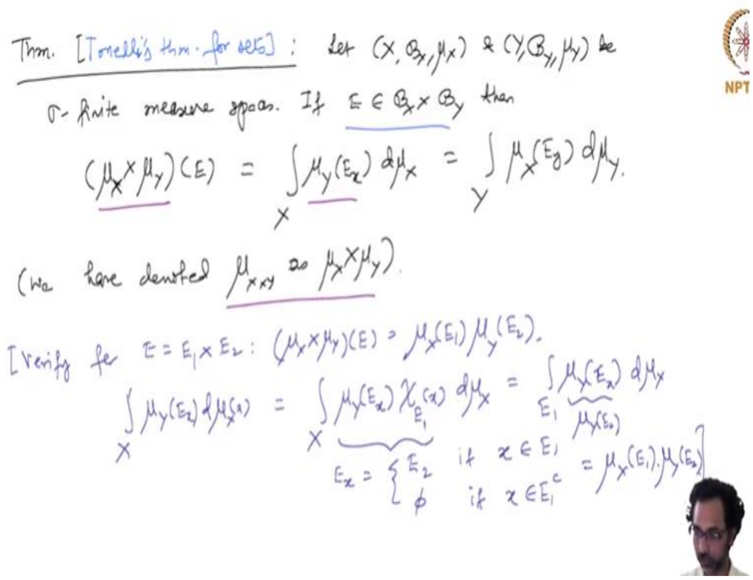

So first theorem in this direction is Tonelli's theorem for sets which says that if we have 2 sigma finite measure spaces X, B x, mu x and Y B, y mu y. And if you take a measurable set E in the sigma algebra B x cross B, y then so here on the left hand side we have denoted a mu x cross y as mu x cross mu y this is the product measure that we constructed earlier. And so the measure of the set E with respect to the product measure is precisely, the integral of the measure of the sections with respect to X take the measure with respect to y and then integrate with this respect to x.

And this is also the same as the measure of mu x E y with respect to mu x and then integral over y with respect to d mu y. So if we verify this for E as a given by a product E1 cross E2 and this should satisfy the product rule that we gave for the premeasure of the when we define the product measure. So let us see whether this satisfies so let us say mu x + mu y of E is then mu x E1 and then multiplies by mu y E2. So this was by definition of the product measure and; now let us see what, are these integrals for this case?

So integral over x mu y E x d mu x so this is equal to integral over x mu y E x Chi E of x E1 of x d mu x so why is this true? Because E x is precisely given by E1 if sorry E2 if X is in E1 and this is equal to the empty set if x belongs to E1 complement. So this is precisely the multiplication by the indicative function and so mu y of E x if x is in is outside E1 this is going to be 0. So this is precisely the integral over E1 of this function mu E mu y E x and integral with respect to the measure mu x.

But now for E1 this is a constant which is mu y of E2 so in the end you will get multiplication of mu x E1 and mu y E2. So we see that this formula checks out when we have a product and we can do the similar thing for first for the integral over y of the sections of E y. So this Tonelli's theorem can be viewed as a generalization of this product rule. So let us see how this is proved?

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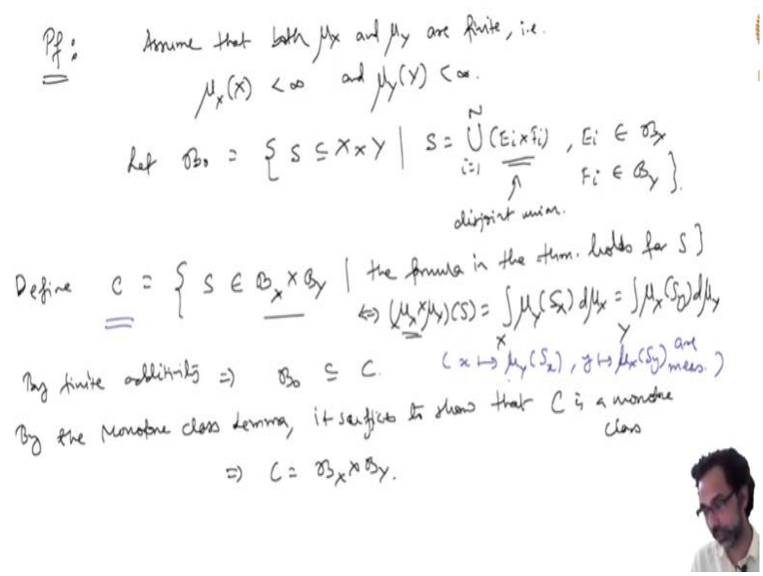

So for the proof we first assume that both mu x and mu y are finite meaning that mu x of the entire space x is finite and mu y of the entire space y is also finite. And then we will pass to sigma finiteness later on. So now let B naught be our Boolean algebra that was defined earlier which was the set of which was the collection of subsets in x cross y such that S is the disjoint union of finitely many it is a finite disjoint union of subsets of the product form where E i and F i belong to the respective Boolean algebras.

So E i belongs to B x Fi belong to B, y so this is the Boolean algebra that were working with we can also assume that this union is a disjoint union. And now let see define C to be the collection of sets in B x cross B, y such that the formula in the theorem holds for S meaning that we have mu x cross mu y of S is equal to the integral of mu y S x over x d mu x and is equal to mu x of S y d mu y.

So we take the collection of all subsets in B x cross B, y such that this is true now by our verification. We have already verified that this is true for this products and by finite additivity we see that B naught is in C. Because if you take a single product then it is in C that we have checked and then we can if we have a finite disjoin union of such products then it will also land up in C because of additivity of the integrals and finite additivity of mu x cross mu y.

So then by the monotone class lemma by the monotone class lemma it is suffices to show that C is a monotone class which will imply that C is the entire space B x cross B, y because the latter is the sigma algebra containing B naught is a sigma algebra generated by B naught. And if we prove that C is a monotone class then they have to be equal by the monotone class lemma. So let us see why? C is a monotone class so let us take and increasing sequence of sets E and in C.

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Let $\{E_n\}_{n\geq 1}^{\infty}$ be an increasing / non-decreasing seq. in C.

and $E = \bigcup_{n\geq 1} E_n$.

To show: $x \overset{f}{\longmapsto} \mu_y(E_x)$ and $y \overset{g}{\longmapsto} \mu_x(E_y)$.

are measurable, and $E \in C$.

First, since $E_n \in C$ $\forall n \geq 1$. then $f_n(x) = \mu_y((E_n)_x)$. is measurable, $g_n(y) = \mu_x((E_n)_y)$ is measurable

Note that f_n increasing pointwise to the fn. $f : x \mapsto \mu_y(E_x)$.

Since $\mu_y\left(\left(\bigcup_{n=1}^{\infty} E_n\right)_x\right) = \mu_y\left(\bigcup_{n\geq 1}^{\infty} (E_n)_x\right) = \lim_{n\to\infty} \mu_y((E_n)_x)$.

(by uMCT).

$\Leftrightarrow$ $f(x) = \lim_{n\to\infty} f_n(x)$ $\Rightarrow$ f is measurable.

So we have E n is increasing or non-decreasing sequence in C and you take the union of all this sets. And we want to show that first of all that these functions that is x to the mu y measure of the section E x and so I should put this below. And the function that takes y to the mu x measure of the section E y they are both measureable and E belong to C. So I should actually go back and say add this in the conclusion of the Tonelli's theorem for the sets.

Because we were integrating mu y E x and mu c E y so it is a necessary condition that this should be measureable so I should here add that x to mu y E x and y to mu x E y are measureable functions on x and y respectively. And then we have this equality similarly for the definition of this monotone that we want to prove it to be monotone class we should also assume here. That this are so this is the collection of all sets such that the formula holds and given that this maps x to mu y Ex and y to mu x E y or rather x y S x and here S y they are measureable.

So here C is the collection of all sets such that first of all that this is measureable these functions are measureable and then that formula holds. So one has to be careful in how we construct these sets and now we have to show that given an increasing sequence of sets in C if you take the union then the maps x to mu y E x and y to mu x E y are measureable and this product formula holds for E.

So first we note that since E n is in C for all n then the maps given by f and x equals mu y of E n x this is a measureable map is measureable. And similarly g n y given by mu x E n y this is also is measurable so these 2 things are measureable because E n is in c and by definition of C these 2 are measureable and further that they also satisfy the integral formula's for E n and for this sections E n y and E n x.

But first let us see what we can do with these sequence of functions so now note that this F n increases point wise to the function f, which takes x to mu y of E x. This is simply because you have that since mu y of the union E n, n = 1 to infinity x this is mu y of the union n = 1 to infinity E n x and this is the limit n tends to infinity of mu y E n x by the upward monotone convergence theorem.

So this means that this sequence of function so this is precisely the limit on the left hand side we have f y and on the f x sorry we have f x and on the right hand side we have f n x and so this is what we wrote that f n increases to point wise to the function f. And so this implies that f is measureable and similarly g can be shown to the measureable g is the function here f is the function here. And so similarly g can be shown to be measureable.

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Similarly g is measurable.

For each $n \geq 1$, $(\mu_X \times \mu_Y)(E_n) = \int_X \mu_Y((E_n)_x)\, d\mu_X$. $(E_n \in C)$

$$\Rightarrow \lim_{n\to\infty} (\mu_X \times \mu_Y)(E_n) = \lim_{n\to\infty} \int_X \mu_Y((E_n)_x)\, d\mu_X.$$

(UCMT) MCT (for fns).

$$\Rightarrow (\mu_X \times \mu_Y)(E) = \int_X \lim_{n\to\infty} \mu_Y((E_n)_x)\, d\mu_X$$

$$\overset{\text{UMCT}}{=} \int_X \mu_Y(E_x)\, d\mu_X.$$

$\Rightarrow E \in C.$

Similarly g is measureable and now we have that for each well let us see for each n we have the rule that mu x cross mu y E n this is equal to the integral over x of the sections mu E n x and the measure mu y of E n x d mu x. Because again E n belongs to C and now we have to take the limit on both sides so the limit as n goes to infinity mu x cross mu y of E n this is equal to the limit as n goes to infinity x mu y E n x d mu x.

Now on the left hand side by again the upward monotone convergence theorem shows that the left hand side is nothing but mu x cross mu y of E. And on the right hand side by the monotone convergence theorem for functions we have this is the integral of the limit n tends to infinity mu y of E n x d mu x. And again by the upward monotone convergence theorem we have this is the mu y of E x d mu x. Similarly one can show for the other variable and so this implies that E belongs to C.

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Similarly if $E_1 \supseteq E_2 \supseteq \ldots$, $E_j \in C$.

then $E = \bigcap_{j=1}^{\infty} E_j \in C$.

Since $\mu_X(X) < \infty$, $\mu_Y(Y) < \infty \Rightarrow (\mu_X \times \mu_Y)(X \times Y) < \infty$.

Use the DMCT to show that $f(y) = \mu_X(E_y)$ is measurable

$$= \lim_{n\to\infty} f_n(y) = \lim_{n\to\infty} \mu_X((E_n)_y)$$

Similarly, $g(x) = \mu_Y(E_x)$ is measurable.

Similarly one can show similarly if E1, E2 is a decreasing sequence E j's in C then the intersection j = 1 to infinity of these E j's belong to C by using that downward monotone convergence theorem by noting that. Since mu x is finite and mu y is finite this implies that mu x cross mu y of x cross y this is finite and on a finite measure space you have the downward monotone convergence for any decreasing sequence. So this implies that so let us put this to be E again.

So now we can use the downward monotone convergence theorem so use the downward monotone convergence theorem. To show that the function f, y which is given by the measure of E y this is measureable because it is a limit of these functions f and y which, is the limit of these measures E n y. Because we have the finite measures spaces so, downward monotone convergence holds for any decreasing sequence.

And so we can show similarly that G similarly g x is so this is given by mu y of E x is measureable and finally let us see if the formula holds.

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$$(\mu_X \times \mu_Y)\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} (\mu_X \times \mu_Y)(E_n) \quad (\text{By DMCT})$$

$$= \lim_{n\to\infty} \int_X \mu_Y((E_n)_x)\, d\mu_X$$

Now, $\mu_Y((E_n)_x) < \mu_Y(Y) \in L^1(X, \mu_X)$

since $\int_X \mu_Y(Y)\, d\mu_X = \mu_X(X)\mu_Y(Y) < \infty$.

Dominated conv. thm.

$\Rightarrow$ $(\mu_X \times \mu_Y)(E) = \int_X \lim_{n\to\infty} \mu_Y((E_n)_x)\, d\mu_Y = \int_X \mu_Y((E_n)_x)\, d\mu_X$

So mu x cross mu y of this intersection E n, n = 1 to infinity this is equal to the limit as n tends to infinity mu x cross mu y of E n again by downward monotone convergence theorem. And this is equal to the limit as n tends to infinity because these are in C so we can use the formula’s for example for x integral over x mu y E n x d mu x. And now we know can no longer use the monotone convergence theorem because these are no longer monotonically increasing functions.

Now we note that these functions mu y E n x this is dominated by mu y of y and this if you view this as a function in x then this is in L1 of x mu x because since the integral over x of mu y, y d mu x this is equal to mu x, x mu y, y and this is finite. So this means that by the dominated convergence theorem we have that the left hand side which is mu x cross mu y of E. This is equal to we can pass the limit inside by the dominated convergence theorem and we have what we want which is mu y E n x.

Similarly one can do it for the other variable and so we have proved it for the case when both x and y are finite measure spaces.

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If X, Y are σ-finite, then $X \times Y = \bigcup_{j=1}^{\infty} (X_j \times Y_j)$

s.t. $\mu_X(X_j) < \infty$ & $\mu_Y(Y_j) < \infty$.

$$\Rightarrow (\mu_X \times \mu_Y)(E) = \lim_{j \to \infty} (\mu_X \times \mu_Y)(E \cap (X_j \times Y_j)) \quad \text{(by uMCT).}$$

$$= \lim_{j \to \infty} \int_X \mu_Y((E \cap (X_j \times Y_j))_x)\, d\mu_X.$$

$$\overset{MCT}{=} \int_X \mu_Y(E_x)\, d\mu_X.$$

And now if x and y are sigma finite then we can write x cross y as the union of pieces X j cross Y j, j = 1 to infinity such that mu x, x j is finite and mu y, y j is finite and then mu x cross mu y of a set E this is equal to mu x cross mu y of E intersection X j cross Y j with the limit this is equal to the limit as j tends to infinity. Again by upward monotone convergence theorem and now this is something in a finite measure space.

So we have that this is the limit as j tends to infinity now the product rule applies well the formula applies the integral formula applies. So let us write for the x integral this is mu y of E intersection X j cross Y j x section d mu x and now we can pass this limit inside by again an application of monotone convergence theorem. So this means that this is d mu y of E d mu E x d mu x so by the repeated applications of upward monotone and downward monotone convergence theorems we get the result this is the Tonelli's theorem for sets.