Measure Theory
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Module No # 12
Lecture No # 65
Tonelli's Theorem

In the last lecture we have seen that a product measure exist on the product sigma algebra and now we are ready to come back to the few theorems of Tonelli's and Fubini on the criteria for interchange of the order of integrations. So for this purpose we first need a technical lemma called the monotone class lemma.

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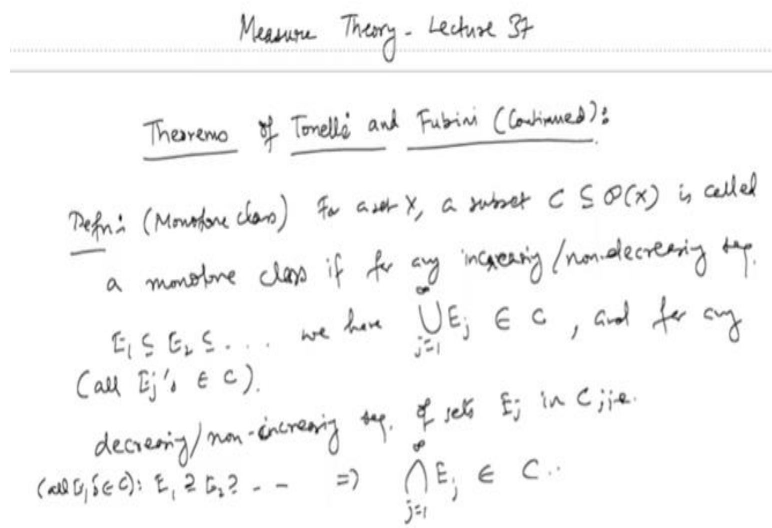

So let me first define what is a monotone class? So for a set X a subset C of the power set of X is called on monotone class. If; for any increasing or non-decreasing sequence E1 subset of E2 and so on so countable sequence of non-decreasing sets. We have the union E j, j = 1 to infinity belongs to c so these are all in C so all E j's are in C. And similarly for any decreasing or non-increasing sequence of sets E j in c which means that we have E1 superset E2 and so on.

And again all these all E j's are in c this implies that the intersection of these E j's also belongs to C. So it is closed under union of sets which are non-decreasing and closed under intersection of sets which are non-increasing. Of course we only take countable intersections and unions.

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Remark: i) An intersection of any family of monotone classes is a monotone class: $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ is a collection of monotone classes in X, then $\bigcap_{\alpha \in A} \mathcal{C}_\alpha$ is a monotone class.

$\Rightarrow$ Given a Boolean algebra $\mathcal{B}_0$, there exist a smallest monotone class containing $\mathcal{B}_0$.

ii) Any σ-algebra is a monotone class.

So let me remark here that an intersection of any family monotone classes is a monotone class which means that if C alpha is a collection of monotone classes in x. Then the intersection of this C alpha's is a monotone class and this implies that given Boolean algebra there exist a smallest monotone class containing it containing. So let me call this Boolean algebra B naught and this smallest monotone class contains B naught meaning that any other monotone class which contains B naught must contains this smallest monotone class and also that any sigma algebra is a monotone class.

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Lemma (Monotone Class Lemma): Given a Boolean algebra $\mathcal{B}_0$, the smallest monotone class containing $\mathcal{B}_0$ coincides with the σ-algebra generated by $\mathcal{B}_0$.

Pf: $\mathcal{C}$ = smallest monotone class containing $\mathcal{B}_0$

$\mathcal{B} = \langle \mathcal{B}_0 \rangle = \sigma$-alg. generated by $\mathcal{B}_0$.

Since $\mathcal{B}$ is a monotone class (containing $\mathcal{B}_0$) $\Rightarrow \mathcal{C} \subseteq \mathcal{B}$.

To show: $\mathcal{B} \subseteq \mathcal{C} \Leftrightarrow \mathcal{C}$ is a σ-alg. containing $\mathcal{B}_0$.

So these 2 remarks will be useful for what we are seeing going to see now which is the monotone class lemma? And this says that given Boolean algebra B naught this smallest monotone class containing B naught coincides or it is equal to with the sigma algebra generated by B naught. So in the case of a Boolean algebra the monotone class generated by B naught and the sigma algebra generated by B naught are the same.

So let us look at the proof of the lemma so suppose that C is the smallest monotone class containing B naught and B be the sigma algebra generated by B naught. And since B is a monotone class and because it is sigma algebra a monotone class by our remark earlier. So this implies that C is a subset of B because C is the smallest monotone class containing B naught and B is a monotone class again containing B naught.

So now it is suffices to show so to show that B is a subset of C and for this it is equivalent to saying that C is a sigma algebra containing B naught. So then we will have equality of C and B so what we will do is the following.

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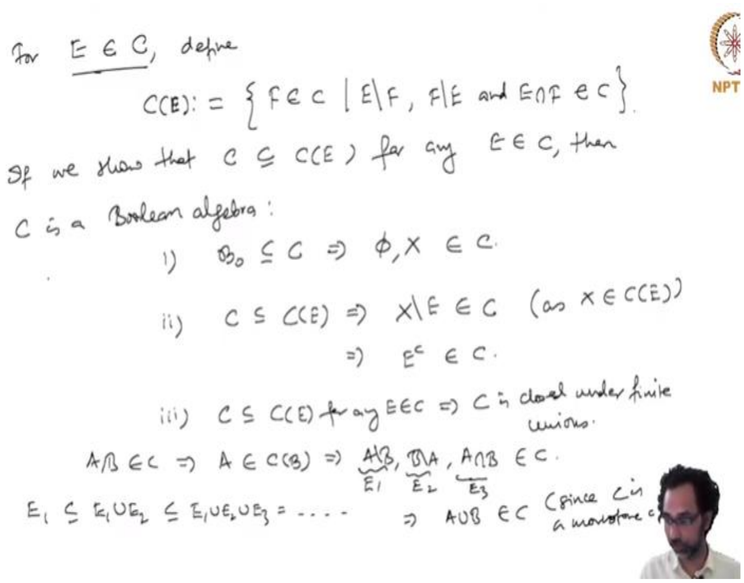

So for E in an element in the monotone class C define C E to be the collection of sets F in C such that E – F, F – E and E intersection F are in all of these are in C. So if we show that C is the subset of C E for any E in C then C is a Boolean algebra why because? First that of course B naught is a subset of C this implies that Phi and X belong to C. Secondly we have C is the subset of C E for any E which means that X – E belong to C.

Because as X belongs to C E this implies that E complement belongs to C so it is disclosed under complements. And third and again if C is a subset of C E for any E in C and this implies that C is closed under finite unions because. For example if you take A and B in C this implies that A belongs to C B which means that A – B, B – A and A intersection B belong to C. And now you can take the increasing sequence let us call this E1 let us call this E2 and let us call this E3.

Then I take the increasing sequence E1 then E1 union E2 and then E1 union E2 union E3 so this is an increasing sequence and then I will just have constant sequence after this one. So this implies that the union of this increasing sequence belongs to C and but the union is precisely A union B since C is a monotone class. So if we show that C is the subset of C for any E then we would have shown that C is a Boolean algebra and from Boolean algebra it is not too difficult to show that C is a sigma algebra.

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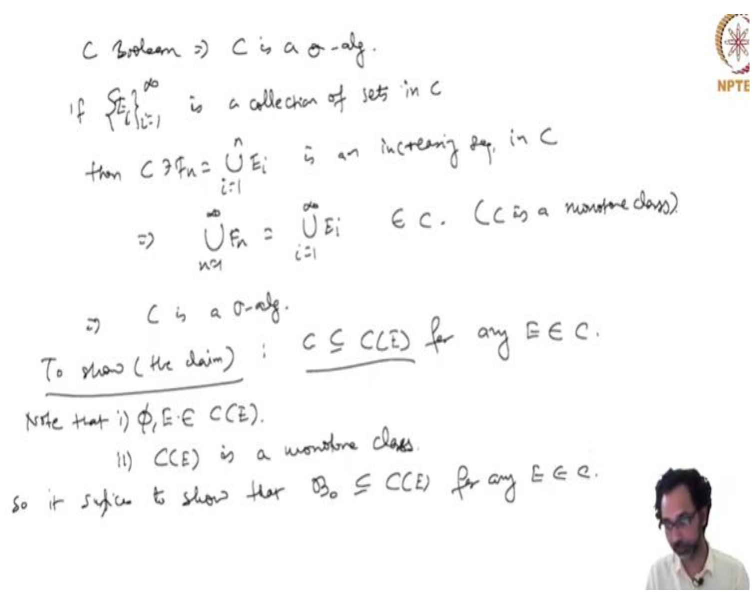

So C Boolean implies C is a sigma algebra because if C is a Boolean algebra it is closed under finite unions and if E1 well E i, i = 1 to infinity is a collection of sets in C then F n i = 1 to, n E i is an increasing sequence in C because all of these are in C due to C being a Boolean algebra. So this is in C and then this implies that the union of all F n which is precisely the union of all, these E i's this is in C because C is a monotone class.

So C is closed under countable unions and because C is already a Boolean algebra then this implies that C is sigma algebra. So we see that if we show this crucial thing that C belongs to C for any C for any E in C then it follows that C is sigma algebra. And so let us show that so to show the claim that C is the subset of C E for any E in C. So first note that both Phi and E so this is the first observation both Phi and E belong to C E and that C E is on monotone class.

Now if I can prove that B naught belongs to C E for any E then we would have shown that C belongs to C because C is a monotone class and C is the smallest monotone class. So it is suffices to show that B naught is a subset of C E for any E in C.

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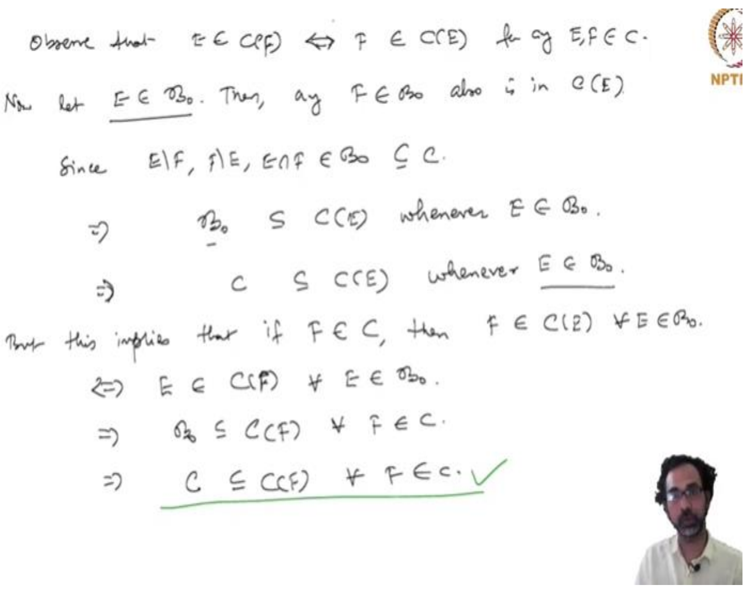

To show this observe that because of definition of C E is the symmetric the relations there namely set minus E – F, F – E and E intersection F. If we interchange the roles of E and F they remain the same so observe that E belongs to C F if and only if F belongs to C E for any E and F in C. So there is this nice relation and now let us take A, E to be in B naught then we have that any F in B naught also is in C E.

So if you fix E in B naught then for any F belongs in B naught is also in C of E because since E – F and F – E and E intersection F they are in B naught and so they belong to C. So for E in B naught any F in B naught is also in C E this means that B naught is a sub collection of C E whenever E is in B naught. This means that C is in C E whenever E is in B naught because this is the smallest monotone class containing B naught and C E is also monotone containing B naught.

So C will be inside C E whenever E is in B naught but this implies that if F is in C now if you take a general F in C then F belongs to C E for all E in B naught which is the same as saying that E belongs to C F for all E belonging to B naught. And this implies that B naught belongs to C F for all F in C and this implies well this is a subset and this implies that C is a subset of C F for all F in C. So this is what are, wanted to prove here instead of E we have F and this is what is our conclusion that C is in C F for all F in C which shows that C is a sigma algebra.

So, this shows the monotone class lemma and now we are going to use this monotone class lemma to prove our theorem of Fubini and Tonelli.