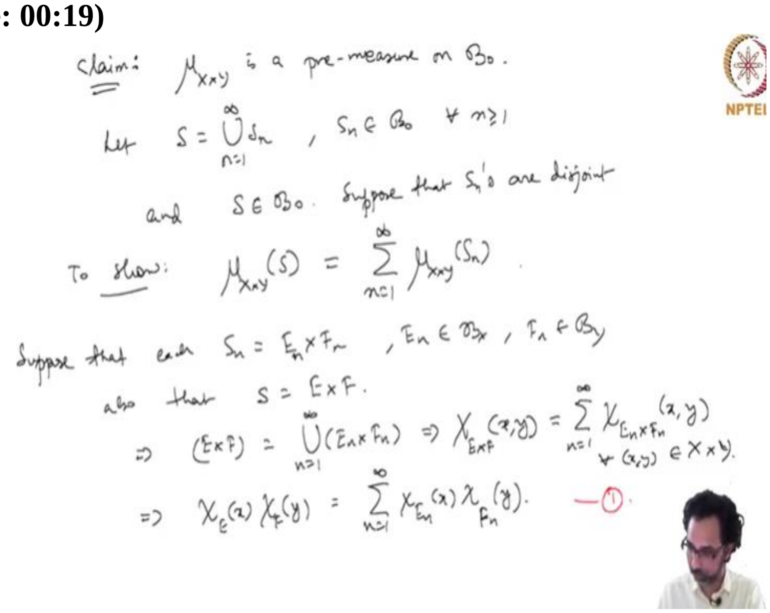
Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Module No # 12
Lecture No # 64
Product measures

(Refer Slide Time: 00:19)



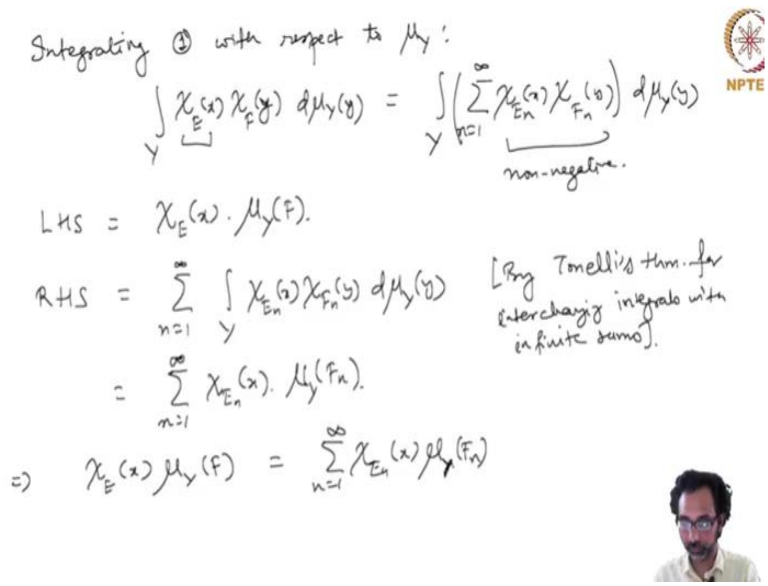
Now we should show that mu naught is a pre-measures so I claim that mu naught sorry not mu naught this is mu x cross y. So mu x cross y is a pre-measure on B naught so to this end let S be a union of S n, n = 1 to infinity such that S n belongs to B naught for all n and S also belongs to B naught. And suppose in that these S n’s are disjoint then we have to show that mu x cross y of S is equal to the sum mu x cross y of S n equal to 1 to infinity.

So let us start with this following case suppose that each S n is of the form a single Cartesian product E n cross F n where E n is in B x and F n is in B, y for each n. And then, also that S is of the form E cross F so since we have that E cross F is equal to the union of E n cross F n. This implies that the indicative function of E cross F x, y is equal to the sum N = 1 to infinity of the indicative functions of E n cross F n at x, y.

But since we have a Cartesian product Chi E cross F is nothing but Chi E x Chi F y and so this is for all x, y in E cross F of course actually X cross Y. So this is true for all X, Y in Cartesian product and we have this point wise inequality. And now we can rewrite this as Chi E x Chi F y

= 1 to infinity Chi E n x Chi F n y. So note that all these Chi E, Chi F, Chi F n are measureable functions in their respectable with respect to these respective sigma algebras. And now we are going to integrate this formula that we have here so let us call this formula 1.

(Refer Slide Time: 04:08)



And then integrating 1 with respect to mu y what do we get so we get integral over y Chi E x Chi F y d mu y. So this is for y and this is equal to the integral of the sum n = I to infinity Chi E n x Chi F n y d mu y. So the first on the left hand side this is simply this is constant and for this integral so Chi E x times the integral of Chi F y. But this is nothing but mu y of F and on the other hand for the RHS we have this equal to by so these are all non-negative functions non-negative.

So by Tonelli's theorem for interchange of summation and integrals we have that this is equal to, n = 1 to infinity integral y Chi E and x Chi F n y d mu y. So this is by Tonelli's theorem for interchanging integrals with infinite sums so but infinity sums of non-negative functions. So we have this and now we can again simplify on the right hand side. So this equal to Chi E n x multiplied by a mu y of F n so then we again have this point wise inequality Chi E x mu y F equals the sum n = 1 to infinity Chi E n x mu y of F n.

(Refer Slide Time: 06:55)

Integrating wrt. μ_X:

$$\Rightarrow \int_X \chi_E(x)\underbrace{\mu_Y}(F)\, d\mu_X(x) = \int_X \left(\sum_{n=1}^{\infty} \chi_{E_n}(x)\,\mu_Y(F_n)\right) d\mu_X(x).$$

$$\Rightarrow \mu_X(E)\cdot\mu_Y(F) = \sum_{n=1}^{\infty} \int_X \chi_{E_n}(x)\,\mu_Y(F_n)\, d\mu_X$$

$$= \sum_{n=1}^{\infty} \mu_X(E_n)\,\mu_Y(F_n).$$

$$\Leftrightarrow \mu_{X\times Y}\overset{\substack{S\\ \shortparallel}}{(E\times F)} = \sum_{n=1}^{\infty} \mu_{X\times Y}\underset{\substack{\shortparallel\\ S_n}}{(E_n\times F_n)}$$

And now we integrate with respect to x integrating with respect to mu x naught integral x Chi e x d there is a mu y F d mu x, X is equal to integral. Again there is a sum n = 1 to infinity Chi E n x mu y of F n d mu x right. And then again we have on the left hand side this is a constant n will come out of the integral and then we only have mu x of E times mu y of F. And on the right hand side again by Tonelli's theorem we can interchange the integral and the summation sign to get integral of Chi E n x mu y F n d mu x.

But this is again nothing but mu x E n mu y F n and since this was on the left hand side we have mu x cross y of E cross F. So we have chosen our S to be of this simple form and on the right hand side we have n = 1 to infinity similarly mu x cross y E n cross F n which was these were all S n's. So we have shown this formula that this formula holds when S n's have this simple form and S has this simple form. And now we can reduce the general case to this case as follows.

(Refer Slide Time: 09:06)

Now suppose in general that $S = \underbrace{\bigcup_{n=1}^{\infty} S_n}_{\text{disjoint}}$, $S_n \in \mathcal{B}_0$

and $S \in \mathcal{B}_0$.

Write $S = \bigcup_{i=1}^{N} (A_i \times B_i)$ (finite disjoint union).

By finite additivity of $\mu_{X \times Y}$; we have

$$\mu_{X \times Y}(S) = \sum_{i=1}^{N} \mu_{X \times Y}\underbrace{(A_i \times B_i)}$$

it suffices to show (by finite additivity) that

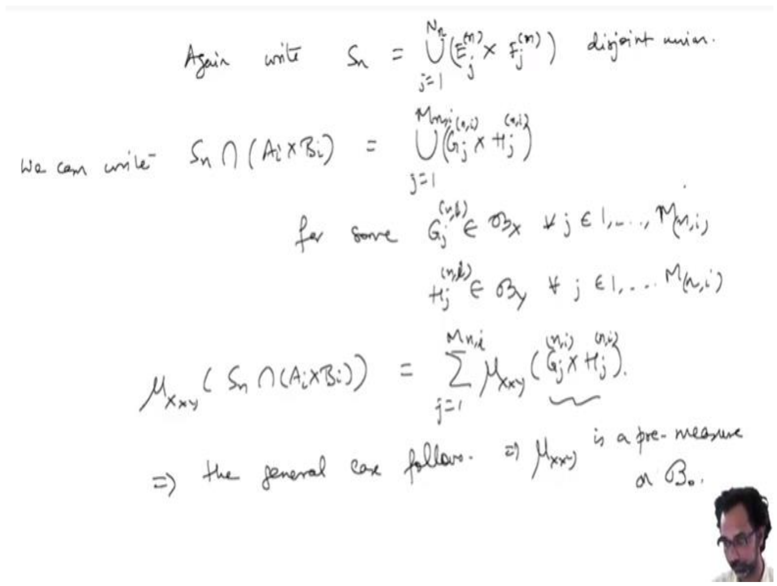
$$\mu_{X \times Y}\underbrace{(A_i \times B_i)} = \sum_{n=1}^{\infty} \mu_{X \times Y}(S_n \cap (A_i \times B_i))$$

Now suppose in general that S is equal to n = 1 to infinity S n = 1 where S n belongs to B X cross B Y for all n this is at disjoint union and for further S belongs to B X cross B Y sorry this belong to B naught and again we have B naught. Now if we write S as a finite disjoint union A i cross B i, i = 1 to N again a finite disjoint union of course this A i cross B i all these A i's and B i's belong to respective sigma algebras B x and B, y.

So by finite additivity of mu x cross y we have that mu x cross y S is equal to this finite sum i = 1 to, n, N Ai cross mu x cross y Ai cross Bi. And now we can take for each N we can take the intersection of S n with these elementary products. So it is suffices to show again by finite additivity that mu x cross y Ai cross Bi is equal to the sum n = 1 to infinity mu x cross y S n intersection with Ai cross Bi.

So we have now reduce it to the case when S is the single elementary sets but on the right hand side this S n may be may not be single elementary sets. But since S n belongs to this Boolean algebra B naught then again we can have a decomposition of S each S n into finitely many pieces.

(Refer Slide Time: 12:27)

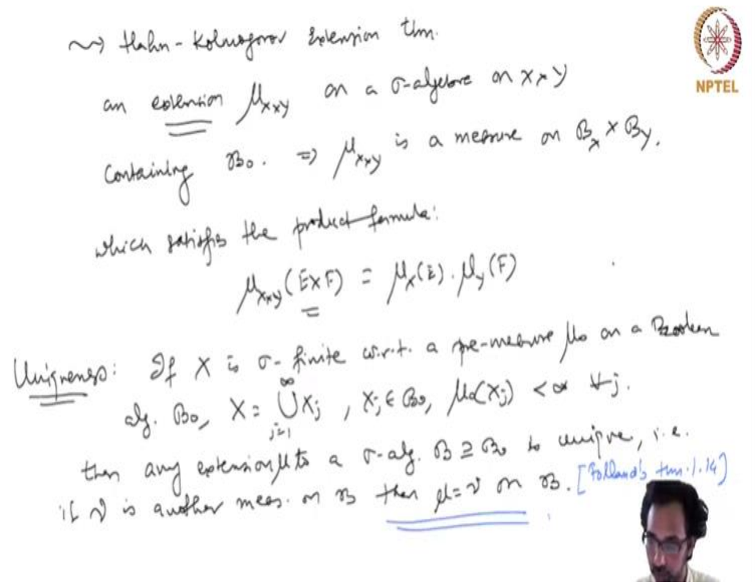


So again right S n = E n i or rather E i n or let me write E j n cross F j n j = 1 to N, n. So this is the number of terms may differ for different S n's but we still have a disjoint union of elements in B naught. And then S n intersection Ai cross Bi is then a disjoint union of the form G j cross H j, j = 1 to sum number M. So it holds or rather we can write for sum G j in B x for all j in 1 to M and H j in B, y for all j in 1 to M.

So we have again use the fact that the intersection and cross product distribute over each other and then we can write sum disjoint union like this. On the other hand then we have that mu x cross y S n intersection Ai cross Bi is equal to the sum j = 1 to M mu x cross y G j cross H j. And so we are again reducing it to elementary sets of this form and so therefore we have. So these are of course dependent on n G j n and so n here G j n's here and they are also dependent on i. So we have write to everywhere n, i so everywhere it depends on n as well as R.

Nevertheless we still have this finite sum of elementary things and so we have reduce it to the previous case where all the S n's where of elementary form in the product form as well as this set S which is the union which is also the product of the form Ai cross Bi. And so we have already shown this to be true and so the general case follows. And so this implies that mu x cross y is a premeasure on B naught.

(Refer Slide Time: 16:34)



So now the Hahn Kolmogorov extension theorem gives us an extension which we have still denote by mu x cross y. On a sigma algebra on x cross y containing B naught and this implies that since B x cross B, y is the smallest sigma algebra containing B naught this means that this mu x cross y this extended mu x + y is a measure on B x cross B, y. And it automatically satisfies which satisfies the product formula that mu x cross y E cross F = mu x E times mu y F because it extends the same because it is extending the mu x cross y the premeasure defined on B naught.

And now for the uniqueness part in fact this is where the sigma finiteness will play a part and this is a general result that if X is sigma finite with respect to a premeasure mu naught on a Boolean algebra B naught which means that X = the union countable union of X j's in the Boolean algebra with mu naught of X j finite for all j. Then any extension to a sigma algebra B containing B naught is unique meaning that if nu is another so let me call it mu here and if nu is another measure on B then mu equal to nu on B.

So I will just give a reference for this general result you can try to prove it yourself but you can also look at Folland's theorem 1.14 which gives a result or which gives a proof of this result. So we also have uniqueness in the sigma finite case.