Measure Theory

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Module No # 13
Lecture No # 63
Theorems of Tonelli and Fubini – interchanging the order of integration for repeated integrals: motivation and discussion of product measure spaces

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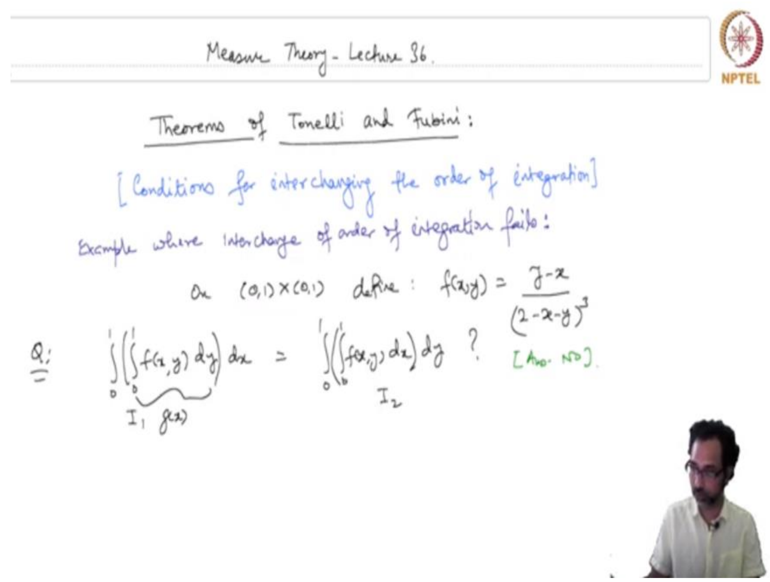

We now come to the theorems of Fubini and Tonelli. So these 2 theorems give us criteria when you can change the order of integrals when you have a multiple repeated integral signs. So one can easily construct examples where this order of integration is not allowed for example if you take as an example where interchange of order of integration fails. So let see so I take on this Cartesian product 0 1 cross 0 1 define the following function f x, y. This is given by y - x over 2 - x - y whole cube.

Now we can try to integrate 0 to 1 f x y d y, d y d x so first integrate with respect to y and then integrate the resulting function so this is a function in x only after integration then you can integrate with respect to x. So the question is whether this is equal to integral with respect to x first f x y d y d x and the resulting function is integrated with respect to dy. So is the are they equal and this answer is no because if you compute for example the this one so let us call this I1 and this I2.

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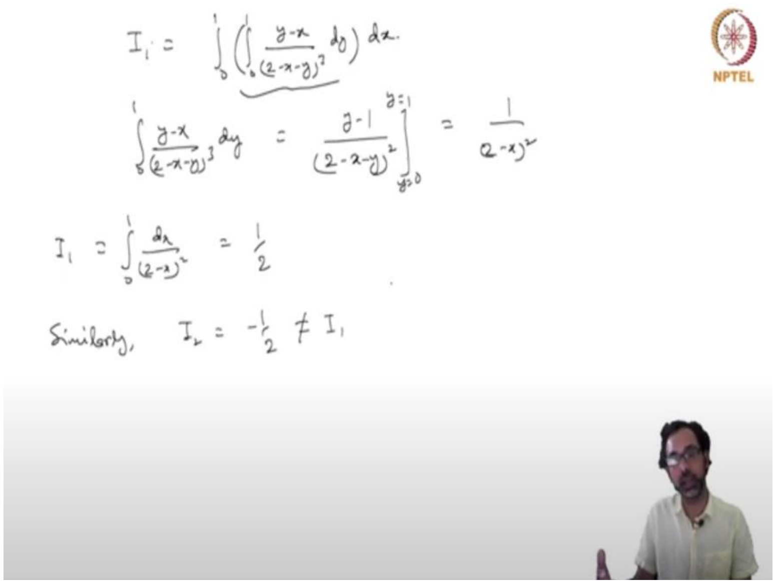

So if you take I1 this is equal to 0 to 1 y - x over 2 - x - y whole cube d y d x. And the interior integral if you want to compute it 2 - x - y whole cube with respect to d y this is equal to y - 1 over 2 - x - y whole square from 0 to 1. And so you will get at 1 you will get 0 so this is y y = 1 y = 0 at 1 this is 0 and at 0 you will get and if you compute this this gives us 1 over 2 - x whole square and now if you want to integrate 1 over 2 – x Whole Squared.

Then one gets plus half and similarly one can compute the other integral I2 but you will get minus half so this is not equal to I1. So interchange of integrals order of integration fails in this case. So the few theorems of Fubini and Tonelli give you criteria when it is allowed to have this change of order of integration and this is not only for the lebesgue measure but also for abstract measure spaces.

So let see what are first we will go we are going to define what are first called product measure spaces and then they will define some product measures on those spaces.

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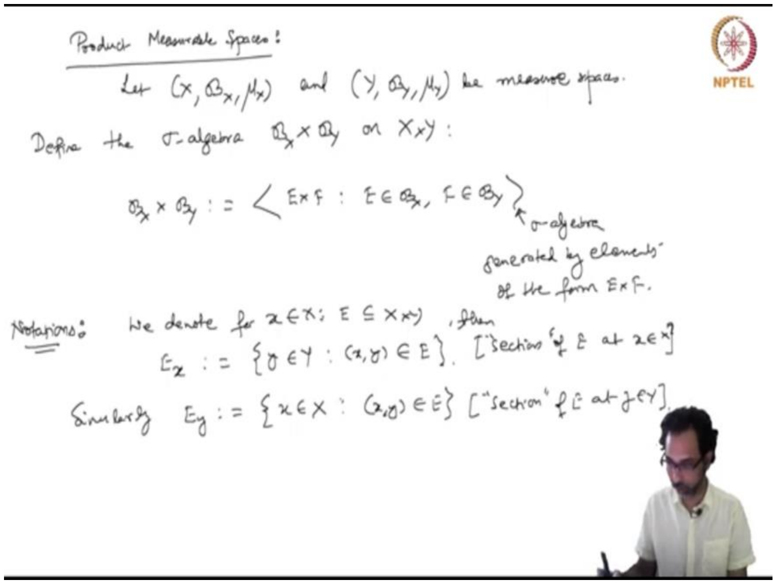

We will define what are called product measurable spaces we will define what are called product measurable spaces. So we consider 2 measure spaces x b x mu x and y B y mu y. So these are measure spaces. So we will define the sigma algebra denoted B x cross B, y on x cross y. So this B x cross B, y is the smallest sigma algebra generated by elements of the form E cross F cartesian product of E and F such that E is in B x and F is in B, y.

So this is the sigma algebra generated by elements of the form E cross F. Now we also use the following notations. So we denote for x in x for set E x which is the set of all y in y such that x. So I am assuming here that E is the subset of so E belongs to B x cross B, y. Then we define this set E x which is the set of all y such that x y belongs to E. So this is called the section of E at x. Similarly we can define E y to be the set of all points x in x such that x y belongs to E and this is the section of E at y.

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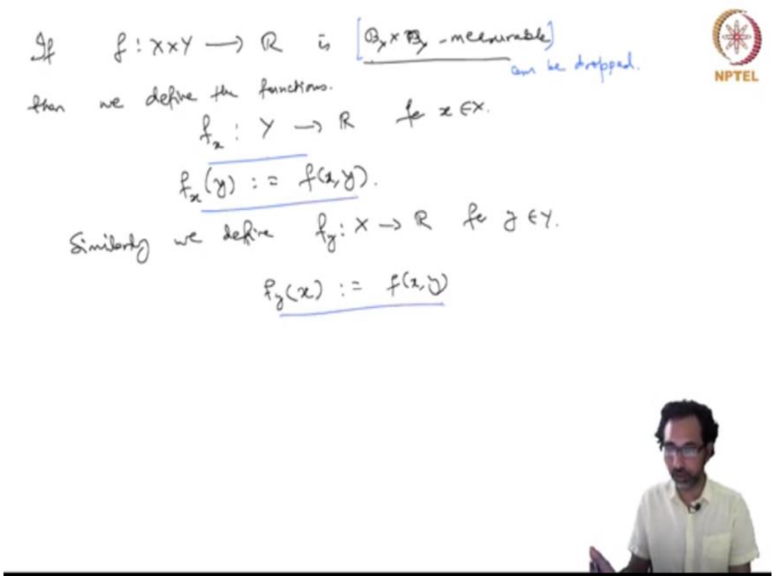

And similarly if F from is a function Cartesian product x cross y say with real values is B x cross B, y measurable this is a real value measurable functions then we define the functions f x this is from the set y to R given by f x y by definition is f of x, y. And similarly we define f y from x to R so this is for x in x and this is for y in y. So f y of x, x is equal to f of x y. So, note that I am assuming this function to be x cross B, y measurable.

But I mean it is not a necessary condition but you can still define these functions even when f x y is not measurable. Similarly here this can just be taken as a subset of x cross y in general and one can still define the section E x and E y. And similarly we can also drop this condition can be dropped and one can still have function which are defined using by fixing one of the variable and then defining it using x using F. So we come to the first proposition considering this product sigma algebra.

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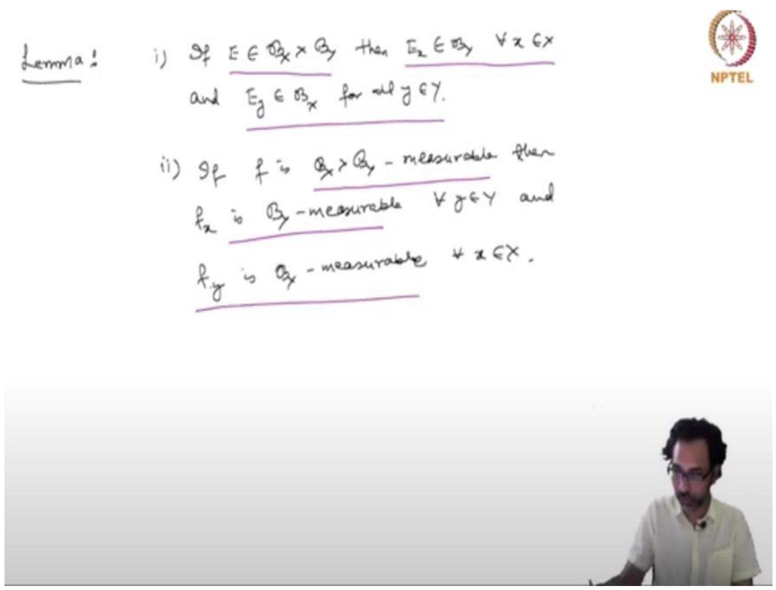

So let us look at the statement of this lemma. So the first part says that if E is a measurable set in B x cross B, y and the section E x belongs to B, y for all x. And then the sections E x belongs to B, y for all x and E y belongs to B x for all y. Similarly if F is B x cross B, y measurable then f x is B, y measurable and f y is B x measurable for every x and y. So let see a proof of this fact proof.

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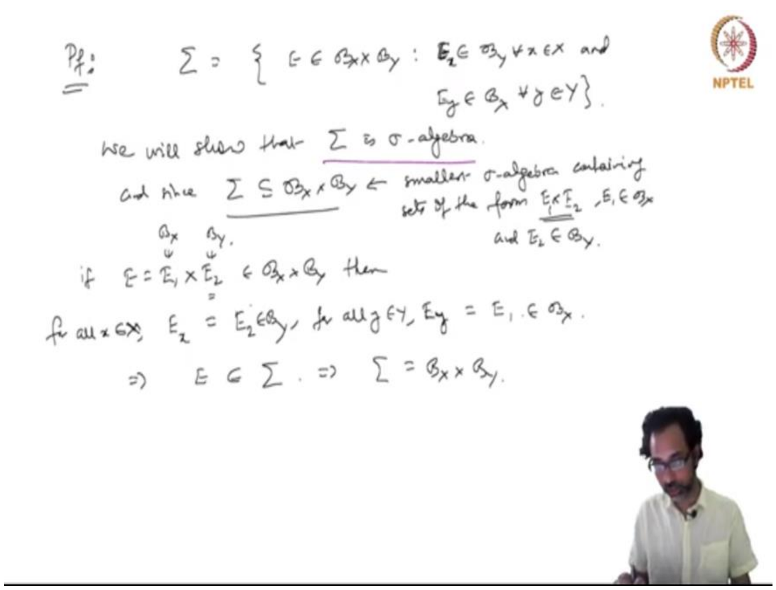

So let me define sigma this is the set of all so let say a cross another E in B x so this are the collection of sets in B x cross B, y such that E x belongs to B x for all x and E y belongs to B E x belongs to B, y sorry and E y belongs to B x for all y in y. So we will show that sigma is a sigma algebra and since sigma is a sub collection of B x cross B, y and this splatter here is the smallest such smallest sigma algebra containing set of the form E cross F or rather E1 cross E2 where E1 is in B x and E2 is in B, y.

But now we will show that these kinds of products E1 cross E2 belong to sigma. So if E = E1 cross E2 belongs to B x cross B, y then for all x in x E x this is equal to simply E2. So this is fixed for all x you will only get E2 and similarly for all y in y E y is E1 and since this belong to B, y and B x respectively. So here I am taking Ex E1 is in Bx this is in Bx and E2 is in B, y. So the sections are simply the other part of the Cartesian product and so this E belongs to this collection sigma.

And since B x cross B, y is the smallest sigma algebra containing the sets of this kind of forms. So this implies that sigma will be equal to B x cross B, y provided we show that sigma is the sigma algebra right. So let us show that sigma is capital sigma is closed under finite union countable unions and as well as compliments. And of course the empty set belongs to sigma and all sets of the form E1 cross E2 belongs to sigma.

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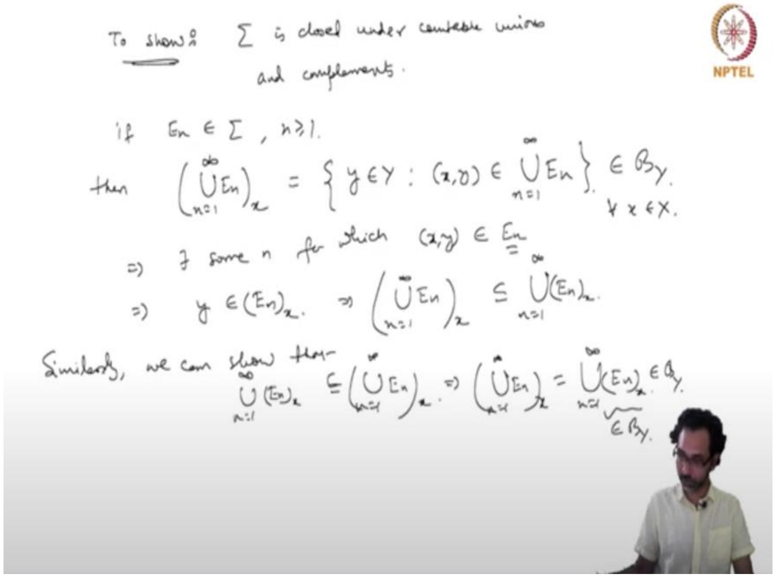

And now we are going to show that to show that the sigma is closed under countable unions and complements. So this is quite easy because En are a collection of sets in sigma then the section of the union n = 1 to infinity the x section is nothing. So this is a set of all points y in y such that x y belongs to the union of this En n = 1 to infinity. And so if x comma y belongs to the union this implies that there exists some n for which x y belongs to En which means that y belongs to En x.

So it is the section of this set En and so this implies that the union En n = 1 to infinity the section of the union is a subset of the union of the sections. And similarly one can show the other inclusion we can show that the union n = 1 to infinity En x is equal is a subset of n equal to 1 to infinity En x. So this implies that the union n = 1 to infinity En x is equal to the union n = 1 to infinity En x.

And so this implies that so all these are all these belong to B, y and B, y is sigma algebra so this means that this is in By and therefore this set belongs to B, y for all x in x. So this means that so similarly one can do it for the y sections so this means that sigma is closed under countable unions.

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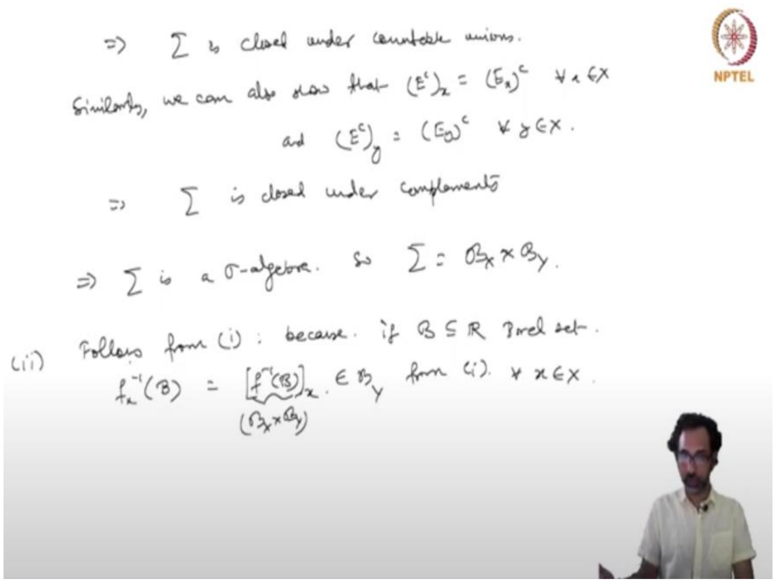

So this means that sigma is closed under countable unions a very similar argument also shows that. Similarly we can also show that the section of the compliment is equal to the compliment of the section for all x in x and similarly for y for all y in x. So this implies that sigma is also closed under compliments and these 2 things taken together mean that sigma is a sigma algebra and this is what we wanted to show to have the equality sigma equals B x cross B, y and so this shows part 1.

For Part 2 this follows from part 1 from 1 because if you take F x inverse of a borel set B so let B be a borel set in R then F x inverse of B is F inverse B of x and because F is measurable this belongs to B, y from part 1. So first of all f inverse B belongs to B x cross B, y and then the x section belongs to B, y from the first part that we have shown and this is for all x in x. And similarly one can do the y version for this and we get the result.

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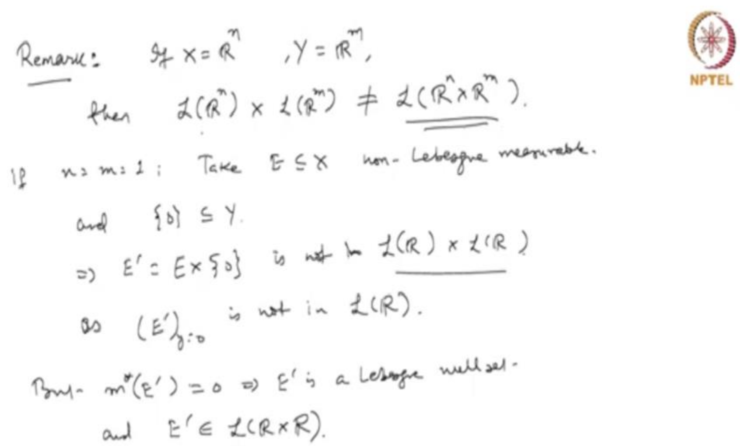

Now let me remark that if you take x to be Rn and y to be Rm then the product of the sigma algebras of Lebesgue measurable sets in Rn and those of Rm this is not equal to the whole lebesgue algebra for Rn cross Rm why because even if n = m = 1 we have elements in the product in the on the right hand side such that the sections do not belong to the Lebesgue algebra. So for example we can take E a subset of x non Lebesgue measurable and take for example the set 0 in y.

So this implies that E cross 0 is not in L Rn cross L Rm because the 0 the section at 0 does not for y because as let me write this as E prime. So if you take the E prime at y equal to 0 then this is not in L Rn. So these are both 1 L R and so this is not in the product algebra but E prime the outer measure of E prime is 0. This means that E prime is Lebesgue null set and so since this is the complete measure space E prime belongs to L R cross R.

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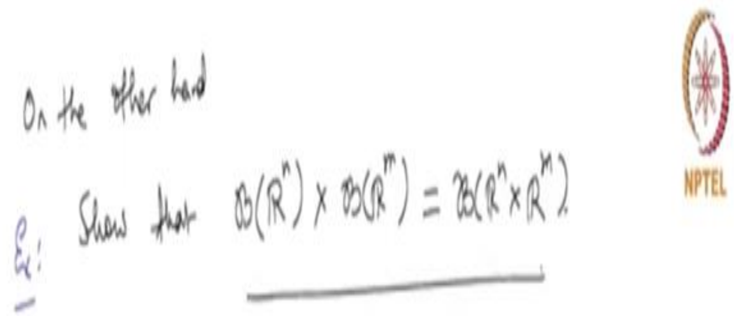

So this shows that there are plenty of other set for the Lebesgue sigma algebra of the cross product if you compare it with the cross product of the Lebesgue algebras. So there are it is far bigger than the left hand side. On the other hand i will prove it I will leave it as an exercise to show that if you take the Borel sigma algebra on Rm and take the cross product with Borel sigma algebra Rm then then the equality holds for the Borel sigma algebra on Rn cross Rm so this hold.

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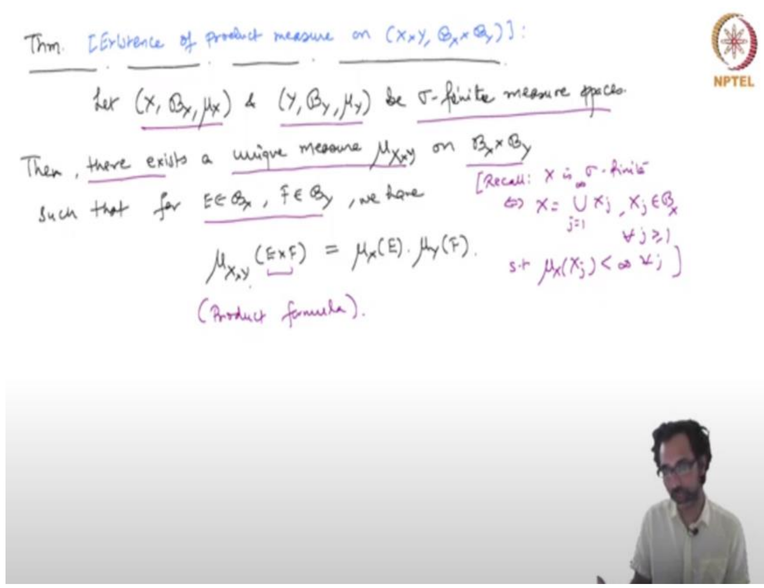

So now let us look at the existence of product measures on this product space x cross y with the product sigma algebra B x cross B, y. So it says that if x B x mu x and y B, y mu y are sigma finite measure spaces so recall that x is sigma finite if and only if x can be written as a countable union of subsets of measurable subsets this is x j in B x for all j such that mu x of x j is finite for all j ok. So this is a sigma finite measure space and we assume that both x and y are sigma finite.

Then there exists a unique measure mu x cross y it is denoted mu x cross y on this product sigma algebra B x cross B, y such that for any E in B x and F in B, y we have the product formula given by mu x cross y E cross F is the product of mu x En mu y F. So this is the product formula that we have seen for elementary measure and it also works for Lebesgue measure and this is the product formula here for Cartesian products sets which are Cartesian products E cross F. So let us see how this can be proved.

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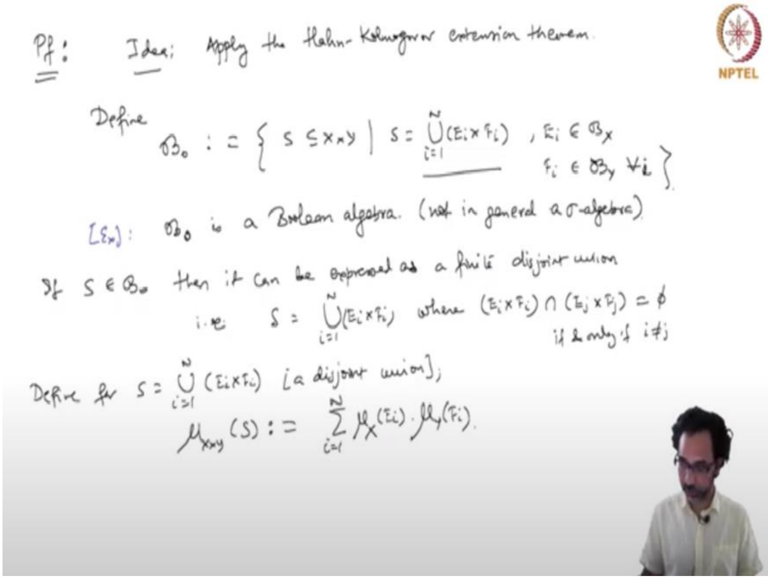

So the strategy the idea is to apply the Hahn Kolmogorov extension theorem. So for this we will need 2 ingredients one is the Boolean algebra and a pre measure on that Boolean algebra. So define B naught to be a subsets of the form S in x cross y such that S is the finite union i = 1 to, n E i cross F i. So these are elementary products sets E i cross F i and then you take the finite union of such where each E i is in B x and F i is in B, y for all i.

So we define it using just an as an analogy with the elementary measures this is how we define the elementary Boolean algebra. And we use the similar approach here to define a Boolean algebra. So now we have to check and leave it as an exercise that B naught is a Boolean algebra. And now once we have this Boolean algebra by the way this is not in general sigma algebra as we have seen for elementary measures elementary sets it is not sigma algebra but it is nevertheless a Boolean algebra it is closed under finite unions and relative components.

So now we need a pre measure now if you take any subset S if S is in B naught. Then S can be expressed it can be expressed as a finite disjoint union this is that S = i = 1 to E i cross F i where the sets E i cross F i intersection E j cross F j is empty if and only if i not equal to j. So this is a disjoint collection just as we did for elementary measures you can also do using similar techniques you can prove that S can be decomposed in disjoint unions.

So once we have disjoint unions we can try to define a pre measure. So let us define for S = i = 1 to, n E i cross F i a disjoint union disjoint union mu x cross y of S is by definition the sum i = 1 to, n mu x E i times mu i F i. So this is the definition and now we have to check a few things which I again will leave as an exercise.

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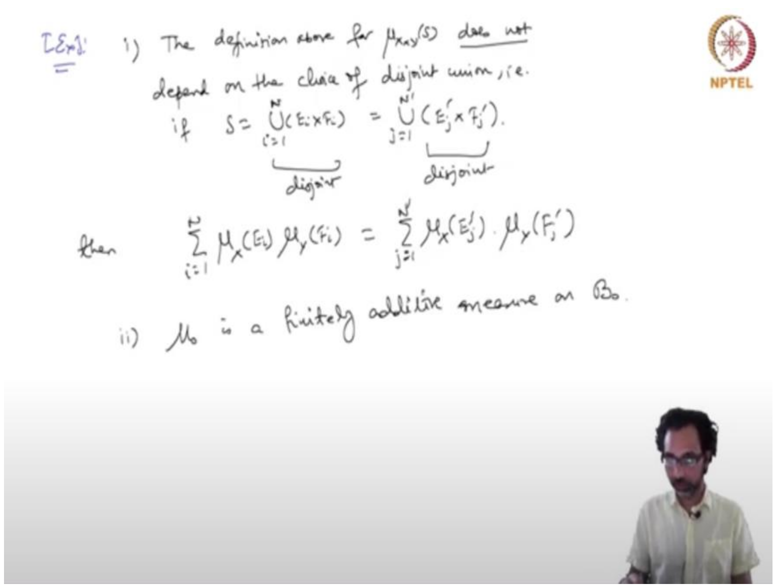

The first is that the definition above for mu x cross y S does not depend on the choice of disjoint unions. Which means that if S = i from 1 to n E i cross F i and it is also equal to j from 1 to n prime E j prime cross F j prime both disjoint unions. So this is disjoint and this is also disjoint. Then we have i = 1 to, n mu x E i mu y F i = j = 1 to n prime mu x E y E j prime multiplied by mu y F j prime.

So this is the first thing one should check that this is not dependent the definition is unambiguous and does not depend on the choice of the decomposition of S into disjoint union. Secondly we should check that mu naught is a finitely additive measure on B naught. So these 2 things are not so difficult to check just by following what we did for elementary measures we can show these 2 results.