Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Module No # 12
Lecture No # 62
Examples of measures constructed using RRT

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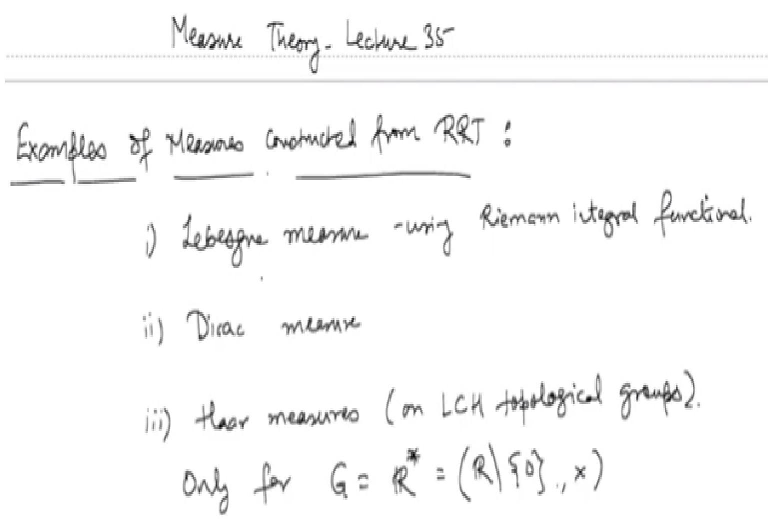

Now we take a look at the examples of measures constructed out of the Riesz representation theorem. So our first example should of course be whether it should answer whether the Lebesgue measure can be constructed out of the Riesz representation theorem. And of course the answer is yes as I mentioned before that the Lebesgue measure can be constructed using the Riemann integral functional.

Secondly we have seen the Dirac measure this can also be constructed using the Riesz representation theorem. And lastly I will just give an example what are called Hoar measures so these are measures on locally compact Hausdorff topological groups. So these are groups which have an underlined space with the topology so the underlying space of the group is the topological space and it is locally compact Hausdorff.

A topological group in a topological group the group operations are continuous with respect to the topology. So a group multiplication and inverse operations are continuous with respect to the group topology and we will construct a not for general groups but only for G = R star which the group of the multiplicative group of non-zero real's. So with the multiplication operation so let us look at how to construct the Lebesgue measure from the Riesz representation theorem.

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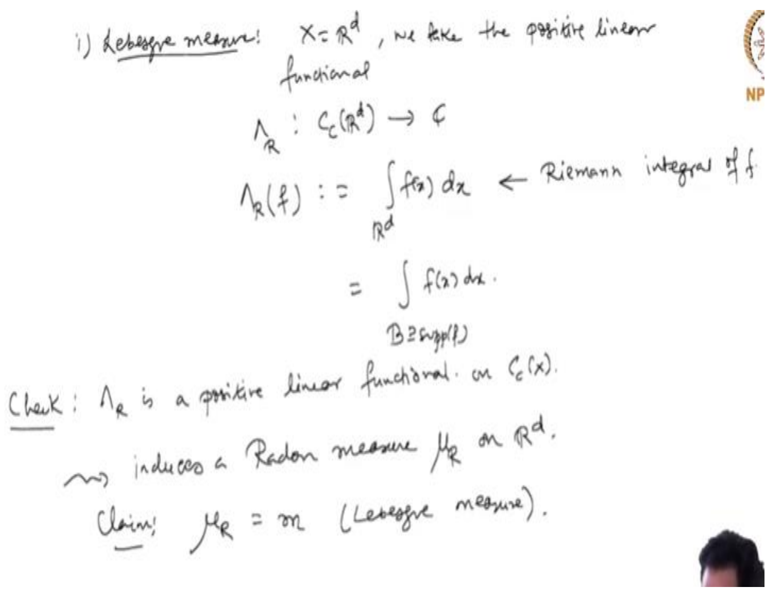

So first is the Lebesgue measure so we have X is Rd and we take the positive linear functional actually positive linear functional which I will denote by lambda R which is from C C Rd to C. And this lambda R is given by is the Riemann integral this is the Riemann integral of, f. So notice that since f has compact support and it is continuous f is Riemann integrable. And you can in fact replace Rd here by some large box that contains the support of f, f x d x.

So of course this lambda R is a positive linear functional as claimed so one can check that lambda R is a positive linear function on C C X. So by the Riesz representation theorem it induces a Radon measure mu R on Rd. And so the claim is that mu R is nothing but our Lebesgue measure on Rd. So to show this it is enough to show that mu R and m agree on open sets because remember that the outer measure that was defined in the Riesz representation theorem was defined using measures of open sets approximated from above.

And so if we have this equality on open sets it is enough to show that mu R = m because then the outer measures will be the same, and then the sigma algebra generated by the Caratheodory measureable sets will be same and so the measures will also be the same.

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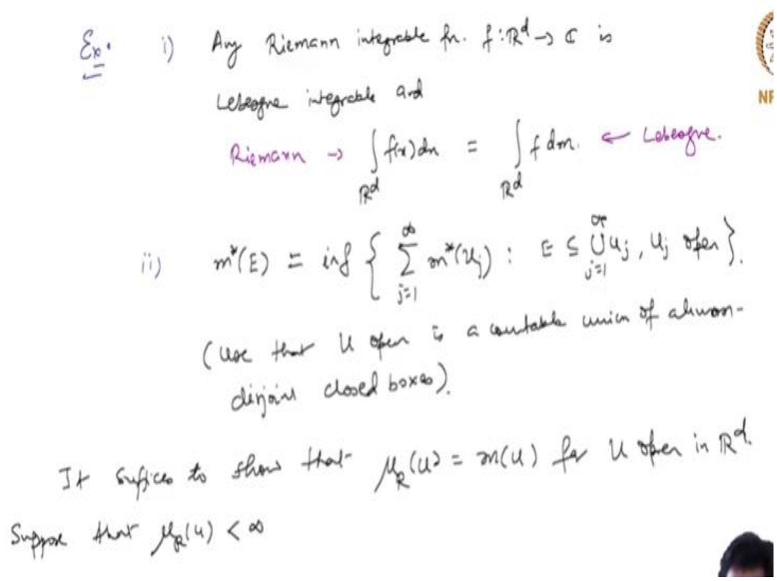

So we need a few results that I will leave as an exercise so the first is that any Riemann integrable function f on Rd is Lebesgue integrable. And in this case we have that the Riemann integrable of fx dx of, f is equal to the Lebesgue integrable of, f with respect to the Lebesgue integrable. So, on the left hand side we have the Riemann integrable and on the right hand side we have the Lebesgue integrable.

So the Lebesgue is can be seen as a generalization of the Riemann integral and if, f is the Riemann integrable then it is Lebesgue integrable and the 2 concepts agree for Riemann integrable functions. So that is the first one second one second point is that m star of E the Lebesgue outer measure can be written as the infimum of sums of m star uj, j = 1 to infinity such that E is the subset of the union of u j's and each of the u j's are open.

So for example you can use that u open is a countable union of boxes of almost disjoint close boxes union of almost disjoint close boxes. So once we have this we have already seen that the outer measure defined in the Riesz representation theorem was defined using the mu's that were defined on open sets using the functional and then we define the outer measure like this. So if the measure from the Riesz representation theorem agrees with the Lebesgue measure for open sets then the outer measures will also agree.

And then the sigma algebra generated of by the Caratheodory measureable sets will also agree and the measures will also agree. So it suffices to show that mu R u equals mu u for u open in Rd. So first suppose that mu R u is finite.

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So let $\epsilon > 0$, and choose $f \prec u$ s.t.

$$\begin{aligned}
\mu_R(u) &\leq \Lambda_R(f) + \epsilon \\
&= \int_{\mathbb{R}^d} f(x)\,dx + \epsilon \\
&= \int_{\mathbb{R}^d} f\,dm + \epsilon. \\
&\leq \int_{\mathbb{R}^d} \chi_u\,dm + \epsilon \qquad (\text{since } f \leq \chi_u). \\
&= m(u) + \epsilon \Rightarrow \mu_R(u) \leq m(u).
\end{aligned}$$

So let epsilon be greater than 0 and choose f less than u such that mu R u is less than or equal to lambda R f + epsilon. So this is because the mu R is the supremum of all such lambda R f so you can choose one such which is close to mu R u. And this is nothing but the integral buy definition of lambda R this is the Riemann integral of, f. And now we know that this is also the Lebesgue integral of, f.

And this is nothing well this is less than or equal to the integral of Chi u d m + epsilon since f is less than or equal to Chi u. And the last is nothing but mu u + epsilon so this means that mu R u is less than or equal to mu.

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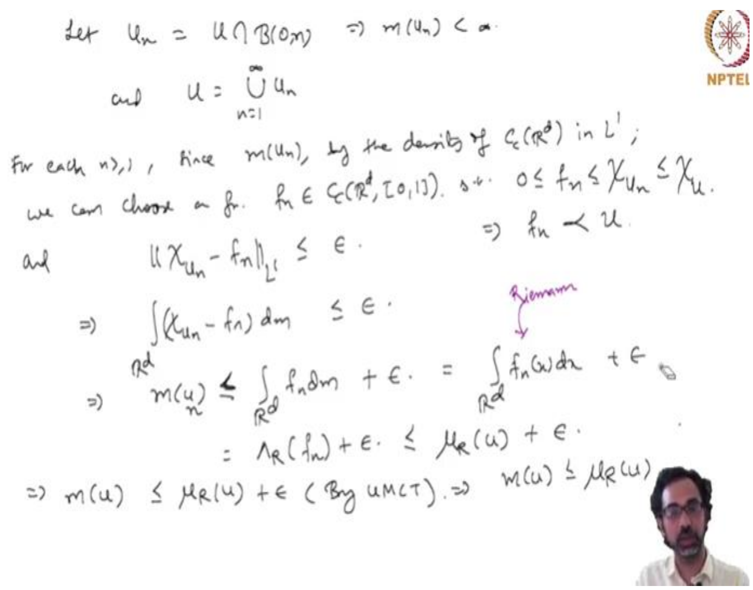

Now for the reverse inclusion let u n be the intersection of u with the open ball of radius n and center 0. So that the measure of u n is finite and u is the union of these is u n's n = 1 to infinity and now for each n. Since the measure is finite by the density of C C Rd in L1 we can choose a function f n in C C Rd with values in 0, 1. Such that 0 is less than or equal to f n is less than or equal to Chi u n and the L1 norm of Chi u n –f n is less or equal to epsilon.

But what does this mean? This means that the integral Chi u n – f n d m is less than or equal to epsilon. And now both are finite so this is m u and you get minus integral Rd f n d m which is less than equal to epsilon. So you can take it on the other side and you will have less than m u is less than or equal to this integral plus epsilon. And now we have seen that Lebesgue integral and Riemann integral coincide for continuous compactly supported functions.

So this is equal to Rd integral over Rd f n x d x where this is now the Riemann integral of, f n + epsilon and on the left hand side you have lambda R f n sorry this is u n lambda R f n + epsilon. And now note that since f n is less than or equal to Chi of u n and Chi of u n is less or equal to Chi of u this means that f n is less than u. So f n as compact support inside u and so this is less than or equal to mu R of u + epsilon.

And now we can take the limit on the left hand side to get mu less than or equal to mu R u + epsilon by upward monotone convergence theorem for the Lebesgue measure. So this implies that mu is less or equal to mu R u and we are done. So we have shown that for open sets so we assume that this was finite and leave it to you as an exercise to show this for the case when this is infinite but then again you can use a limiting argument as I have done here.

So I will leave it to you as an exercise so this shows that the Lebesgue measure can be constructed out of the Riemann integral functional in this way we can see the Lebesgue integration as a completion of Riemann integration.

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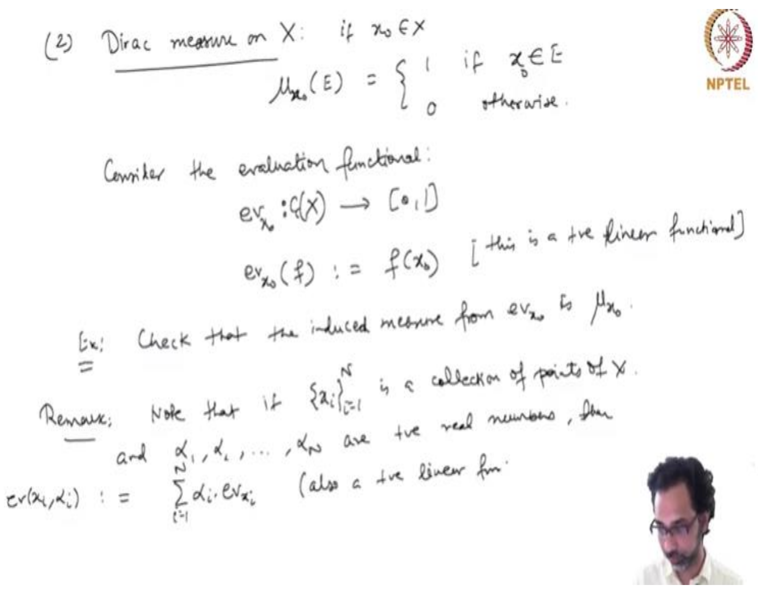

Another example is the construction on the Dirac measure on X we have already seen what it is? So if we fix a point X naught in X then mu X naught of a set E is equal to 1 if X naught belongs to E and 0 otherwise. So I will construct measure out of the Riesz representation theorem by considering the evaluation functional which is given by. So I will write E V X naught from X to 0, 1 and this is by definition sorry this is over C C X to 0, 1.

And if you apply a function with continuous function with compact support then by definition this is f of X naught this is why it is called evaluation because it is evaluating the function f at the point X naught. And as an exercise I will leave it you to check that the induced measure from the evaluation functional. So the first one has to check that this is a positive linear functional this is quite straight forward because if f is positive then f X naught is positive and linearity is obvious because f + g is defined using the point wise addition so it is positive so it is also linear.

And now we have to check that the induced measure from e v x naught is indeed the Dirac measure mu X naught. So I leave it to you as an exercise and as a remark note that if xi, I = 1 to N is a collection of points of x and alpha 1 alpha 2 alpha n are positive real numbers. Then the finite linear combination alpha i e v x i so let me denote e v x i alpha i this is the finite linear combination of the evaluation functional with coefficient alpha i.

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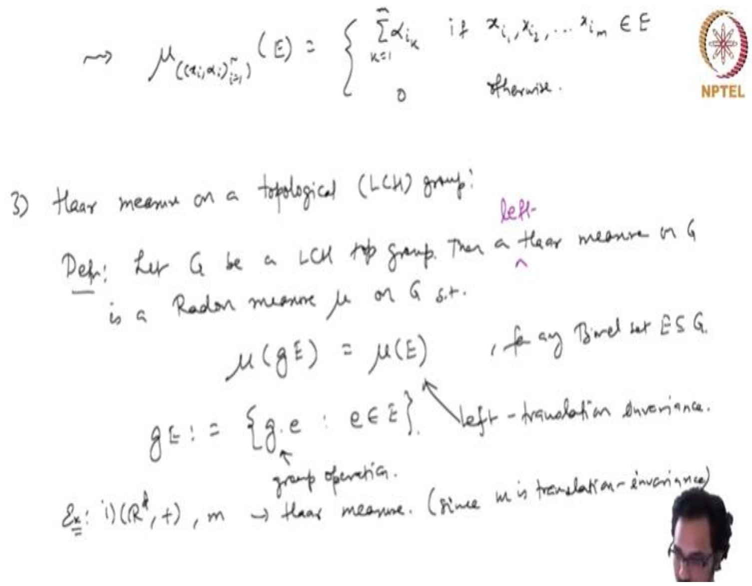

This is also a positive linear functional so the induced measure over these points i = 1 to, n is given by the following formula is equal to sum of alpha i. If so it is sum of alpha i k, k = 1 to m if x i1, x i2 up to x in belong to E and 0 otherwise. So the maximal set of points that belong to E that contribute to this measure but it will be 0 otherwise. So it is generalization of the Dirac measure and you can construct this using the Riesz representation theorem.

Now the third example is the Hoar measure on topological group well we also wanted to be LCH locally compact Hausdorff. So a Hoar measure so let G be a locally compact Hausdorff topological group. Then a Hoar measure on G is a Radon measure on G such that so Radon measure so let me denote it by mu such that the measure of g E is equal to measure of E where g E. So this is for any Borel set E inside g and g E by definition the set of points g times E such that E belongs to E.

So here we are just using the group operation so this is called the left translation invariance property invariance. And in fact this is what is called a left Hoar measure so you can also have the right Hoar measure where g acts on the right and you could also ask for both but it is rare. So as an example we have Rd m where Rd is taken as the topological group with the addition with the Lebesgue measure m this is a Hoar measure.

Because m is translation invariant so this group is a (()) (21:54) groups so left and right actions are both the same. So this give you a Hoar measure so that is one example.

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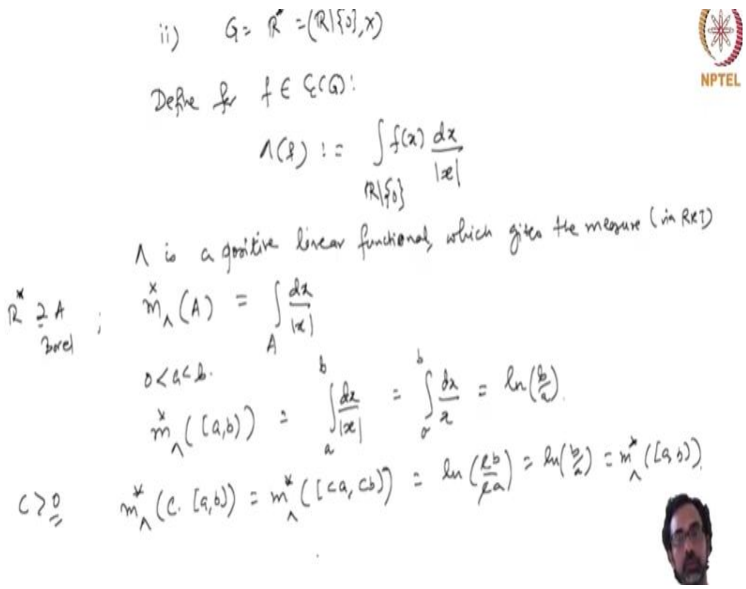

Another example is of G is the multiplicative group of real's so this is R – 0 with the multiplication operation. And if you define for f in C C G lambda f as the integral of, f over R star fx dx over mod x. So this is over set R – 0 and this is the usual we can take it as the Lebesgue integral and if you define this functional in this way then it is again is a positive linear functional which gives the measure via the Riesz representation theorem given by.

So let me denote m cross as a measure induced by lambda so m cross lambda of A where A is a Borel subset of R is given by the integral over A of dx over mod x. So for example m cross lambda of an interval a, b. So a – 0 so of course is a subset of R star s so a will not contain in 0 anyway so a, b – 0 if a is if this interval contains 0 then we remove it from it from this interval. And so this will be the integral well let me take it both positive so that it is easier.

So then there is no zero inside and so this is simply the integral from a to b of dx over mod x which is the same as integral over a to be dx over x and this is l n b / a. And it is a left invariant because m star lambda of c times this interval a, b for example this is nothing but the measure of the interval c a, c b and this is nothing but l n of c b / c a. Here again I am taking C to be positive strictly positive so then you will have you can also take it to this strictly negative but then you will have l n c b over c a.

So this is again l n b over a with is m star lambda a, b so this is with respect to the multiplication group operation this is invariant. So this is the Hoar measure on the group of multiplicative non-negative real’s