Measure Theory
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Module No # 13
Lecture No # 61
Riesz Representation theorem – Complete statement and proof – Part 2

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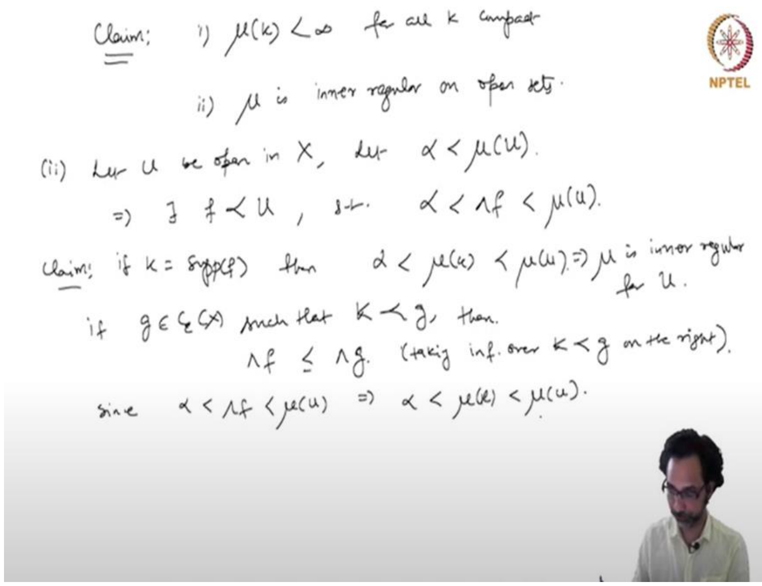

So we claim that first is that mu k is finite for all k compact this follows immediately because it is the infimum of lambda f and all these lambda f's are finite. So mu k is finite. Second is that mu is the inner regular on open sets. So how do we show this? So let u be open in x and suppose that let alpha be chosen says that it is less than mu u. So this implies that there exists f less than u such that alpha less than lambda f less than mu u because mu u is the supremum of all such f's all such lambda f.

So if you choose any alpha less than mu u there will be something lying in between mu u and alpha. So now we have choo we have that alpha less than lambda f less than mu u. Now we will show that if k is chosen to be the support of, f which is compact then so let me write it alpha less than mu k less than mu u. So this will show that actually I am proving the second part. So I claim again here that if we choose k to be the compact set support of, f then alpha less than mu k less than mu and this implies that mu is inner regular for u right.

So why should this be true if g is a continuous function with compact support such that k is less than g then lambda f is less than or equal to lambda g because f is 0 outside of k and g is 1 on k and f takes the maximum value 1. So this means that lambda f is less than or equal to lambda g. So since alpha is less than lambda f is less than mu u this implies that alpha is less than mu k is less than mu u by taking infimum over k less than g on the right hand side. So we showed that mu is inner regular on the open set.

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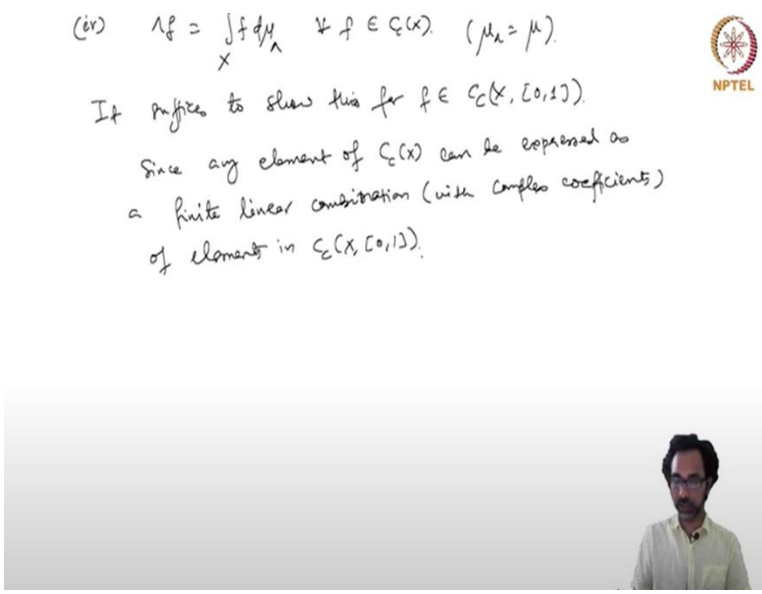

Now we come to the last claim which was the lambda f is actually given by the integral of, f with respect to this d mu lambda. So of course, our mu lambda is the mu that we wrote. And to show this we note that it suffices to show this for f in C C x but with range in 0, 1. Because since C C x is a linear span rather any element of C C x can be expressed as a finite linear combination so with complex coefficients of elements in C C x 0 1.

Because f is compactly supported and continuous it has a minimum and a maximum and so we can partition the range into various pieces of size 1. And then we can simply write any element of C C x as a finite linear combination of elements in C C x with range 0 1.

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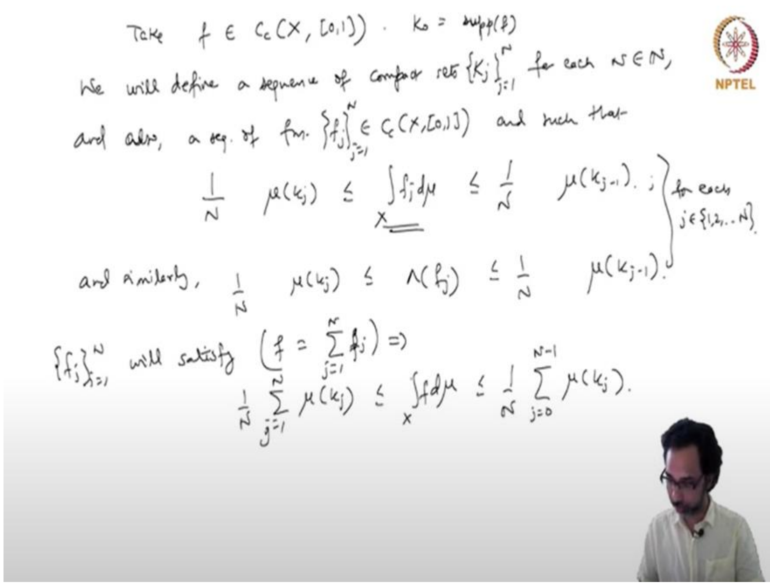

So let us take a function take a function f which is continuous compactly supported with values in the interval 0, 1. So we denote k 0 to the support of, f which is compact. Now we will define a sequence of a compact sets k j, j = 1 to N for each N in N and we will define and also a sequence of functions f j which will be again in c 0 x 0 1. So this is again a sequence j = 1 to N and such that we will have 1 over N sum j = 1 to N mu of k j less than or equal to integral f j d mu over x which is less than or equal to 1 over N j = 0 to N - 1 mu of k j -1.

And similarly we will have the same inequality for lambda of, f j mu k j less than or equal to i lambda of, f j and this is less than or equal to j = 0 to N - 1 mu of k j - 1. And this sequence of function f j, j = 1 to N will satisfy there is no summation here yet. These are inequalities that we have and this f j will satisfy that f the function f that we started out with is the sum of all these f j. And so this would imply that 1 over N if you sum this inequality so this is for each j for each j in 1 up to n.

So if you sum all these inequalities all these are non-negative numbers. So you can sum all these inequalities and you will get on the left hand side some of mu k j, j = 1 to N and in the middle you will get for the first one f d mu because f is the sum of, f j's. On the right hand side you will get sum j = 0 to N – 1 mu of k j.

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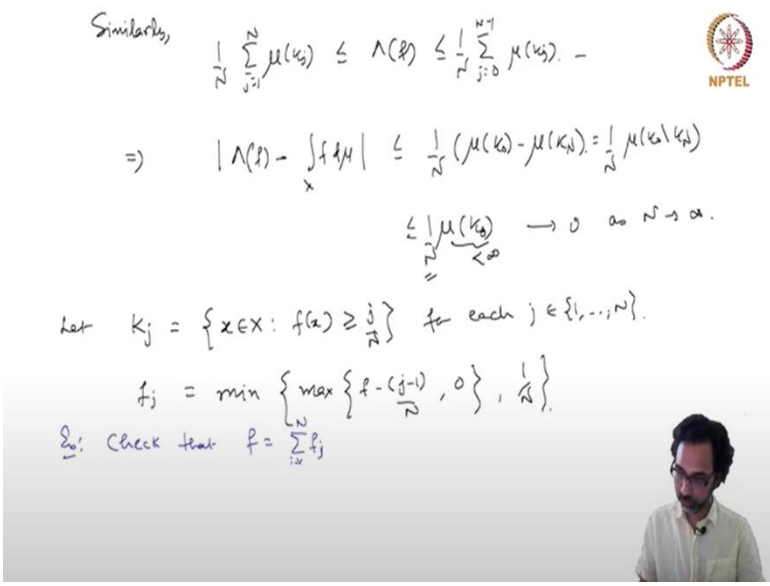

And similarly we also have the same inequality lambda of, f. So you will have mu k j, j = 1 to N on the left lambda f and then on the right you have the sum j = to 0 to N - 1 mu k j. So this implies that the difference lambda f – f d mu has absolute value less than or equal to 1 over N mu of k 0 - mu of k N from just by subtracting this inequality with the one for integral f d mu. So since this, is equal to 1 over N mu of k 0 - k N and this is less than or equal to 1 over N mu of support of, f which is k 0.

So this part is finite and N is arbitrary so this goes to 0 as N goes to infinity which implies that these two things are equal. So now we have to just build up our sequence of compact sets k j and our sequence of function f j such that they satisfy these inequalities, as well as that f is the sum over all these f j's f is the sum over all these f j's. So we will use the following formula so let k j be the set of points in x such that f x is greater than or equal to j over N for each j between 1 to N.

And f j is equal to the minimum of the maximum of f - j - 1 over N comma 0 you take the maximum of these 2 things and the minimum of this value and 1 over n. So one has to check that so this is an exercise check that f = sum over f j, j = 1 to N. So let us see what these f j look like.

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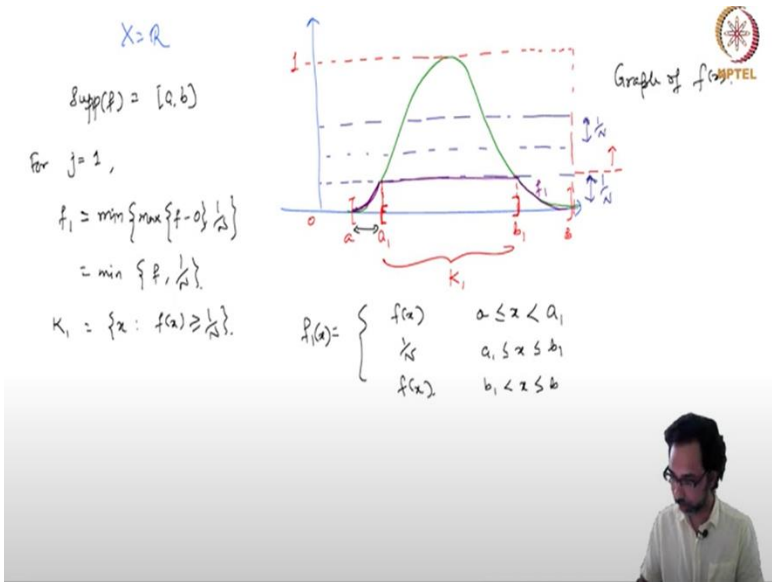

So let suppose that x is our real line and we have nice function belt shaped function with the compact support given by the interval a, to b and the height is exactly equal to 1. So let see what we have done first of all we have divided this range into parts of length 1 over N. So this is 1 over N and so we have divided the whole thing like this so each has width or height 1 over N. So now let us see what is our f1 so j = 1 for j = 1 f1 is the minimum of the maximum of f - j - 1 over n.

So it is 0 comma 1 over N. So this is nothing but the minimum of, f comma 1 over N. So now let see what is k1 this is equal to the set of all point such that f x is greater than or equal to 1 over n which means that k1 is nothing but this set her. So if you take this point and if you take this point then I will denote that N point as a1 and b1 and in within this interval a1 and b1 the value of f x is greater than or equal to this value 1 over N.

So f x has values upwards of 1 over n so this is our k1. So now let us see what kind of graph f1 has so f1 is equal to f1 of x. So now we can have the partition if a0 sorry if a less than or equal to x less than or equal to or less than a1. Then we have that f1 is less than 1 over N. So f1 is simply f so this is f x. Now if a1 less than or equal to x less than or equal to b1 then it will have so in this region it is having this part in the graph.

But in the interval a1 to b1 it is going to be constant because we are taking the minimum of f and 1 / N, n y 1 / N and on k1 the value of f is greater than 1 / N. So within in k1 it is going to be constant it takes the constant value 1 over N and then again so let me write it here is 1 over N. And again if b1 less than x less than or equal to b then again it is f of x so this part of the graph is also part of f1. So this is f1.

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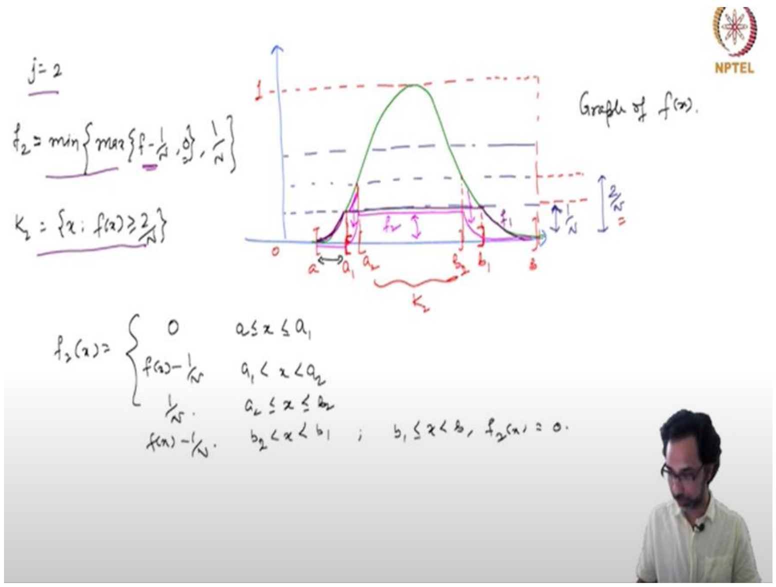

Now let us look at what kind of graph f2 has it will be having a similar kind of graph. So now let us take j = 2 and now we have f2 is the minimum now the maximum value of f – 1 / N comma 0 and the minimum value of this value and 1/N. And we have k 2 is the set of all points such that f x is greater than or equal to 2 / N. So let us see what kind of set we get for k 2. So this height is 2 / N and so for k 2 we will get this interval a2 b2 and this is k 2.

And now f2 if we take f2 x then between 0 less than or equal to x less than or equal to a1 what do we get? So it is the maximum value of f – 1 / N comma 0. But f – 1 / N is negative in this region because f is less than or equal to 1 / N. And so this is going to be simply 0 and we take the minimum of 0 1 / N so this is again 0. So this is between a, and a1. Let us take between a1 and a2. Between a1 and a2 the function has a value greater than 1 / N but less than 2 / N.

So if you take the maximum of f – 1 / N comma 0 you will get f – 1 / N. And f – 1 / N is still greater than 1 / N because it is less than or equal or 2 / N. So you will get f - f x – 1 / N. So you are taking this part so let me take another color. So you are taking this part and you are pushing it down 1 / N. So this part is 0 this part is just the copy of the one in green part above. And then we will have again 1 / N in this part for k 2.

So for a2 less than or equal to x less than or equal to b2 it will have simply 1 / N. And then again if b1 less than x less than b2 less than x less than b1 you will get f x – 1 / N. And if b1 less than equal to x less than b we will get f 2x to be simply 0. So here in magenta color we will have up to here 1 / N. So this is 1 / N. And then again this part is dropped up to here and then you will get a 0. So this is the graph for f2. So in fact one can write down a general formula for f j.

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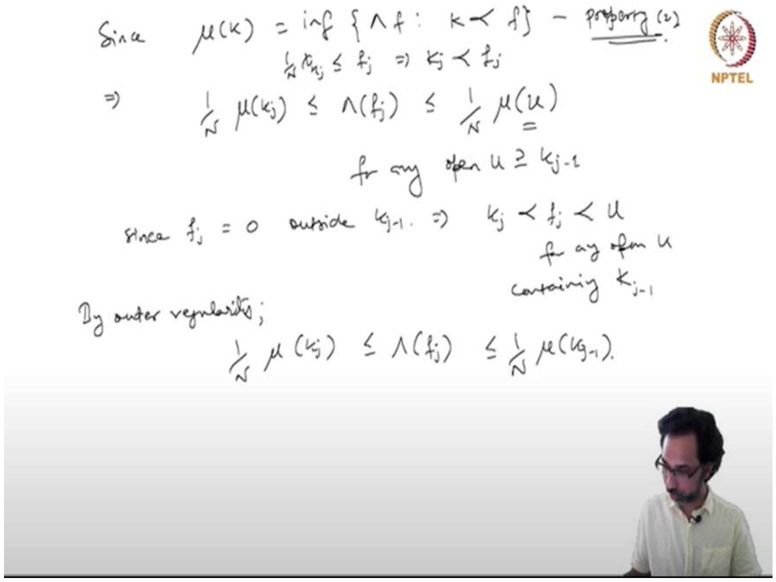

So f j x for j in 1 2 up to N we have f j x is equal to this is equal to 0 if x does not belong to k j - 1 this is going to be 0. If x belongs to k j - 1 - k j notices that k j is the subset of k j - 1 for h j. And for this case we will have f - f x - j - 1 over n and finally if x belongs to k j then it is going to be 1 over n. So this implies that we have the point wise inequality 1 over N Chi k j less than or equal to f j less than or equal to 1 over N Chi k j - 1.

And we also have that f is equal summation of f j, j = 1 to N. And if we integrate this inequality we get 1 over N mu k j less than or equal to integral f j d mu less than or equal to 1 over N mu k j - 1. And now we are going to use the fact that mu is radon to derive a similar formula for lambda fj. So since mu k is the infimum of all the lambda f such that k less than f. This implies that 1 over N mu k j is less than or equal to f j lambda f j this is because 1 over N Chi k j less than or equal to fj.

So this means that f j or rather k j is less than f j. So this implies 1 over N mu k j is less than or equal to lambda f j. And on the other hand we have that lambda f j is less than or equal to 1 over N mu u for any open u containing k j - 1 this is because simply because f j since f j is equal to 0 outside k j - 1. So this implies that k j less than f j less than u for any open u containing k j - 1. Now by outer regularity so this was property 2 that we proved before property 2.

And now by outer regularity we can approximate mu k j - 1 with respect to the open set and so we can take the infimum on the right hand side. So we get 1 over N k mu k j is less than or equal to lambda f j is less than or equal to 1 over N mu k j - 1 and this is what we wanted to show in order to derive our result. So let us go back to these 2 results so one was this when you sum up all the inequalities.

So this is this was the first inequality that we needed 1 over N mu k j less than or equal to integral f j d mu less than equal to 1 over N mu k j - 1. And similarly the same inequality for lambda in N place of the Lebesgue integral. And now you sum it up and then you derive that lambda f - integral f d mu is less than or equal to 1 over N mu k naught which goes to 0 as N goes to infinity. So this finishes the proof of the fourth part which said that lambda f is equal to integral of, f d mu. Now we have to show one more thing which is uniqueness.

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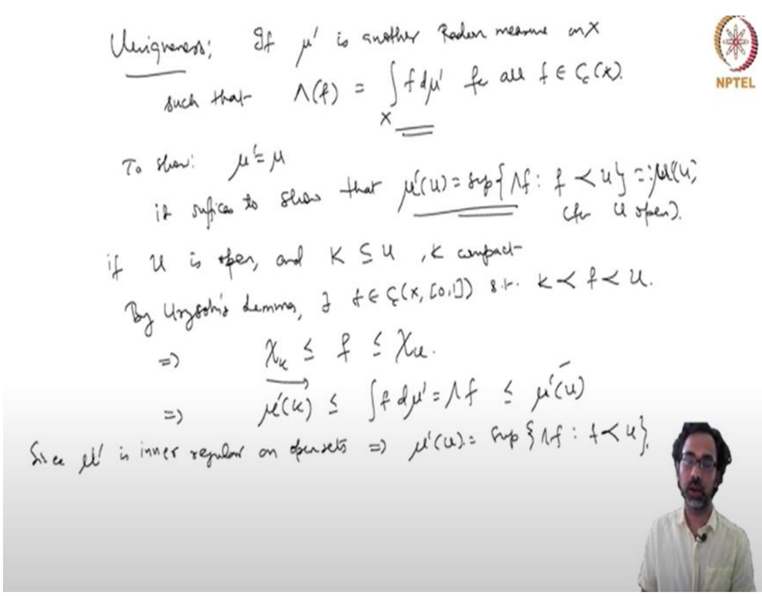

So suppose that mu prime is another Radon measure on x such that lambda f is equal to integral f x d mu prime for all f in C C x. But then we have so it is suffices to show that mu prime equals to mu it suffices to show that mu prime u is given by the same formula of the supremum of lambda f such that f is less than u and this was by definition our mu of u. So then we just have to show this result and to show this result we proceed as follows.

So if u is open so this is for u open. If u is open and k a subset of u a compact subset and I am going to use the inner regularity of mu prime because it is a radon measure in regularity for open sets. So, now by Urysohn's lemma there exists f in C C x 0 1 such that k less than f less than u. And this implies that because lambda f is given by the integral this implies that mu k if you integrate the indicative functions.

So let me write it in terms of indicative function so Chi k is less than or equal to f less than or equal to Chi u. And now if you integrate you get mu prime k less than or equal to integral f d mu prime which is equal to lambda f and this is less than or equal to mu prime of u. And now since mu prime is inner regular on open sets this implies that mu prime u is the supremum of all such lambda f such that f is less than u.

Because the left hand side converges to mu prime u as you take an increasing sequence of compact subsets of u. So we also prove we have also proven uniqueness and this finishes the proof of the Riesz representation theorem. And now in the next lecture we will see what kind of examples of measure we can construct out of the Riesz representation theorem.