Measure Theory
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Module No # 12
Lecture No # 60
Riesz Representation theorem – Complete statement and proof – Part I

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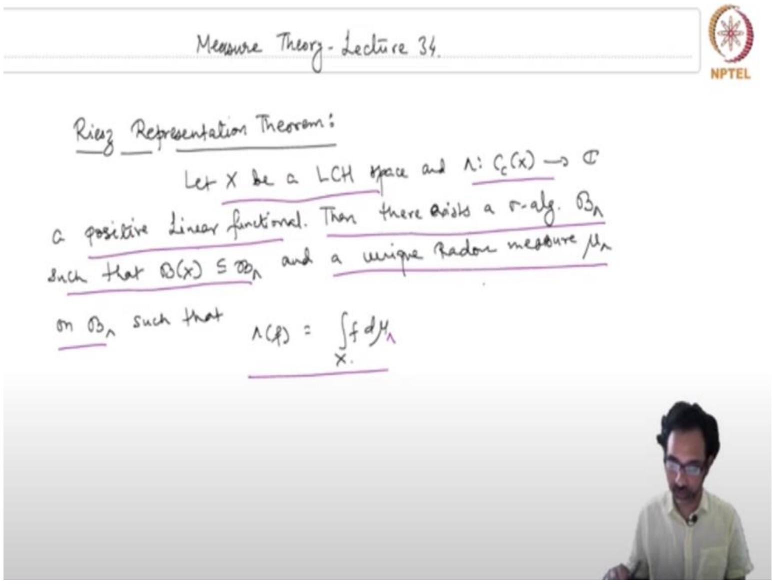

Now let us come back to the statement of the Riesz representation of theorem. So recall that it said that if you have a levelly compact Hausdorff space and positive linear functional on the space of continuous compactly supported functions on x. Then there exist a sigma algebra B lambda which contains all the Borel sets. So the borel sigma algebra lies inside the B lambda. And a unique radon measure mu lambda defined on B lambda such that the linear functional lambda f is given by the integration against this measure mu lambda.

And I did not give this additional nice properties that mu lambda satisfies. So let me give it now.

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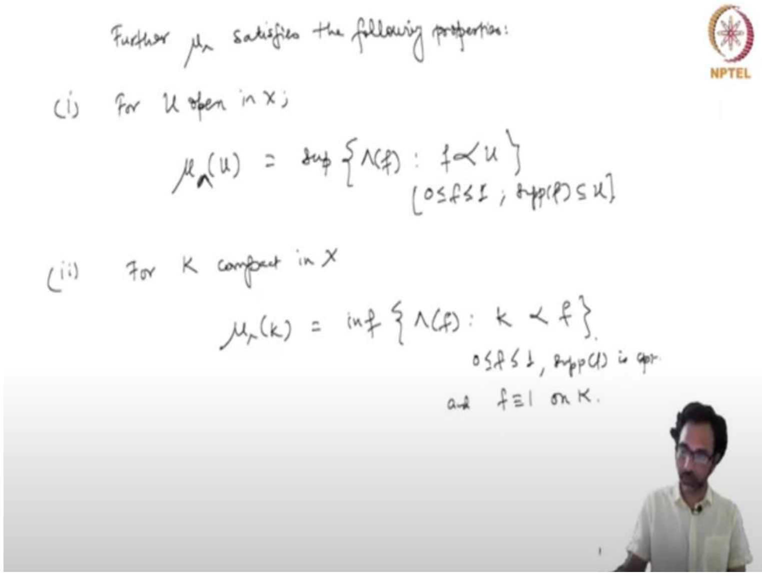

So, further mu lambda satisfies the following properties. So the first property is that for u open in x mu lambda of u is the supremum of lambda f such that f is less than u and the second property is that for, k compact in x. We have that mu lambda of k is the infimum of lambda f such that k is less than f. So remember that this is again the notation that we used before so this means that 0 less than f less than equal to 1 and the support of f is inside the u. This is what f less than u means.

And k less than f means that again 0 less than equal to f less than 1 support of, f is compact. And f is identically equal to 1 on k. So in addition to being Radon measure this mu lambda satisfies these 2 properties where its value on open sets is given by the supremum of this lambda f. Where f ranges over functions which have compact support inside u and which are taking values only between 0 and 1.

And similarly for k compact the value mu lambda k is given by the infimums of these values lambda f. Where, f range is over all function which has compact support and range between 0 and 1 which are identically equal to 1 on k. So let us look at the proof of the Reisz representation theorem. Now there are many proofs available and I have decided to follow the proof in Folland’s book.

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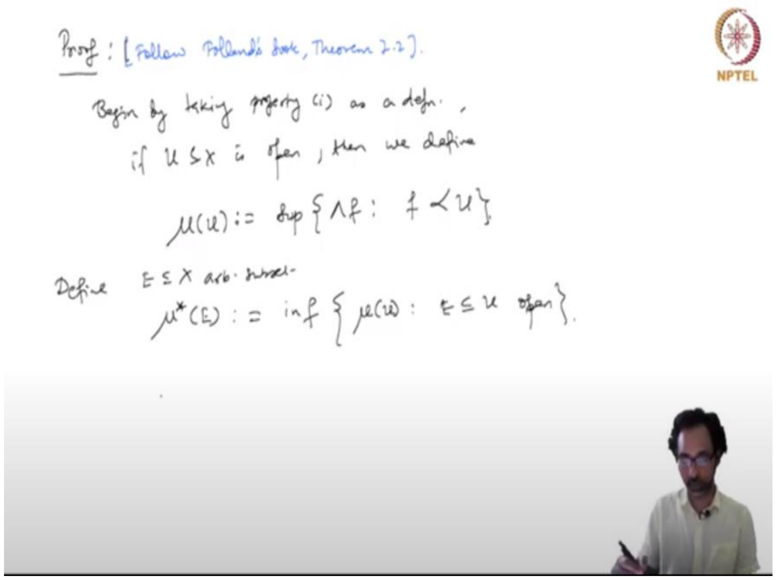

So for the proof I will follow Folland's book which I found the proof I found to be quite nice so I will follow that. So this is theorem 7.2 in Folland's book. So now we begin by taking property 1 above as a definition. So this means that if u is open then we define mu u to be the supremum of lambda f such that f is less than u. So we take this as a definition for the measure mu u for u open and now we define mu star of E.

So now E is an arbitrary subset of x and then we can define mu star of E as the infimum of mu u such that E is contained in u which is open. And we claim several things so this proof will require several steps which shall prove various properties of mu and mu star. So let us list them one by one and try to prove them.

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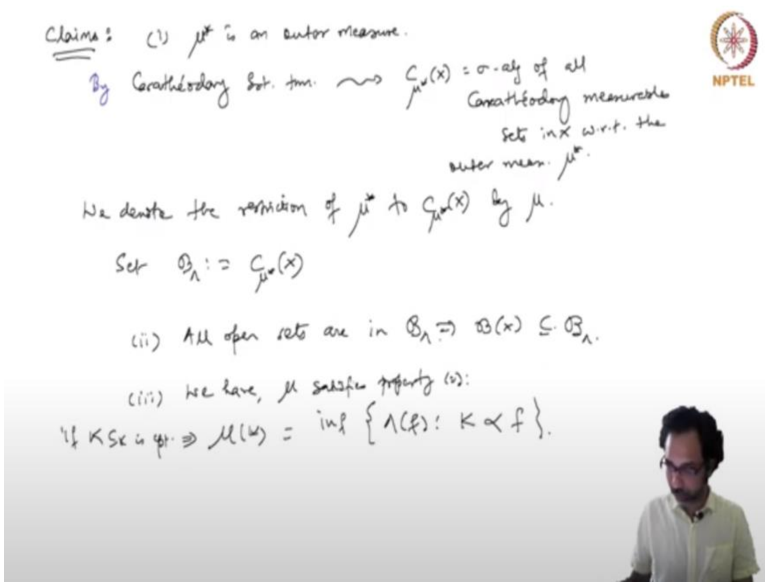

So we make the following claims first one is that mu star is an outer measure. So the mu star that we just define is an outer measure. So if mu star is an outer measure by caratheodary measurable extension theorem we get c mu star x which is the sigma algebra of all caratheodary measurable sets in x with respect to the outer measure mu star. Now the restriction of mu star so we denote the restriction of mu star to c mu star x this is a measure and this is denoted by mu.

And this is again due to the caratheodary extension theorem. Now we set our B lambda to be precisely this sigma algebra of caratheodary measurable sets. Now the second one is that all open sets are in B lambda and this implies that the borel sigma algebra sets inside B lambda because the borel sigma algebra is the smallest sigma algebra containing all open sets and B lambda contains all open sets.

So the second claim is that all open sets are in our caratheodary measurable. Thirdly we have so because all borel sets are inside B lambda we have that all compact sets are also measurable caratheodary measurable. And we have that mu satisfies property 2 which is that mu k so if k is compact this implies that mu k is the infimum of lambda f such that k is less than lambda. So this was part of the additional property of mu on compact sets.

And what we are claiming here is that once we define property 1 once we take property 1 as a definition then property 2 follows for compact sets.

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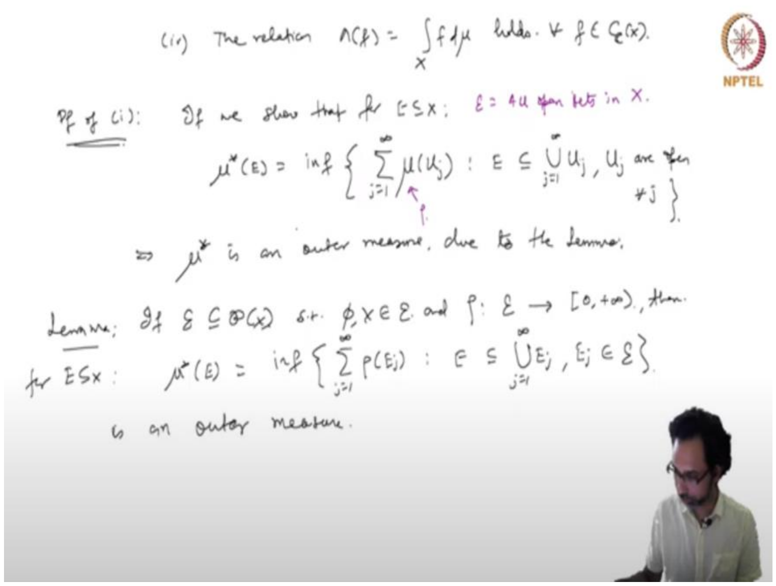

And lastly that the relation lambda f equals f d mu holds for all f in C C x. So let us prove these claims one by one. So proof of the first part so we have to show that mu star is an outer measure. So if we show that for any arbitrary subset E we have that mu star E is the infimum of these sums mu u j, j = 1 to infinity such that E is covered by the union of u j's j = 1 to infinity and all u j's are open.

So if we show this then this implies that mu star is an outer measure due to the lemma that, we once stated which was for due to the following lemma let me write it down. Due to the lemma so this lemma said that if E is a collection of subsets of x such that phi so the empty set and the entire set x both belong to E. And a rho is a map from E to 0, infinity. Then mu star of E so for E in x mu star E given by infimum of rho E j, j= 1 to infinity such that E is a subset of the union of E j's j = 1 to infinity.

And E j belongs to this collection E this is an outer measure. And this was proved I think I left it as an exercise but I said that the proof is exactly as you would prove that the lebesgue outer measure is an outer measure. So I left this proof as an exercise. So here our rho is in fact this mu this is our rho and our E is the collection of all open sets in E open sets in x. So then if you show that this equality holds then mu star will be automatically an outer measure.

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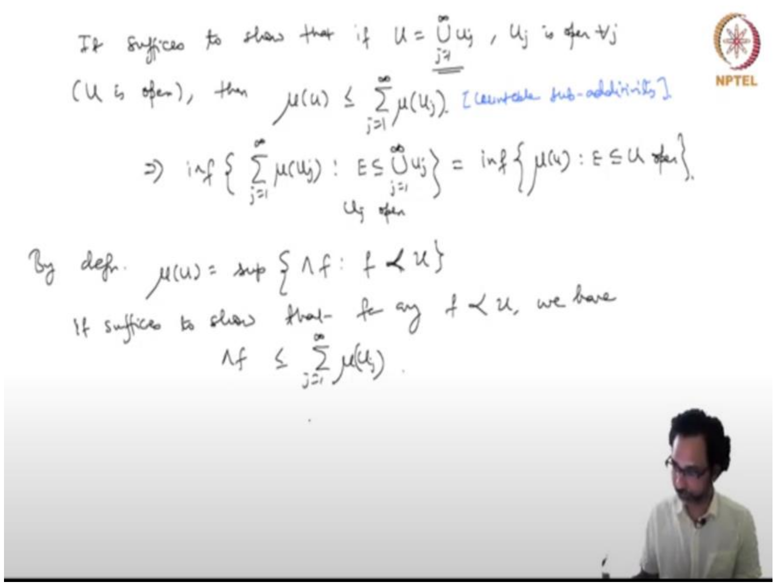

So to show this note that it suffices to show that if mu if u is open u j infinity such that u j is open for all j. So u itself is open u is open then mu u is less than or equal to this sum j = 1 to infinity mu u j. So this will imply that the infimum over these sums j = 1 to infinity mu u j such that e is covered by u j's u j open and this is equal to the infimum of mu u such that E is covered by a single open set u.

So we just have to show that if u is open and given by a countable union of open sets then it satisfies this is the countable sub additivity property. So let so note that by definition so by definition we have that mu u is the supremum of lambda f such that f is in u. So in turn it suffices to show that for any f less than u we have lambda f is less than or equal to this sum mu u j, j = 1 to infinity. So then we can take this supremum on the left and we will get mu u. So how do we show this?

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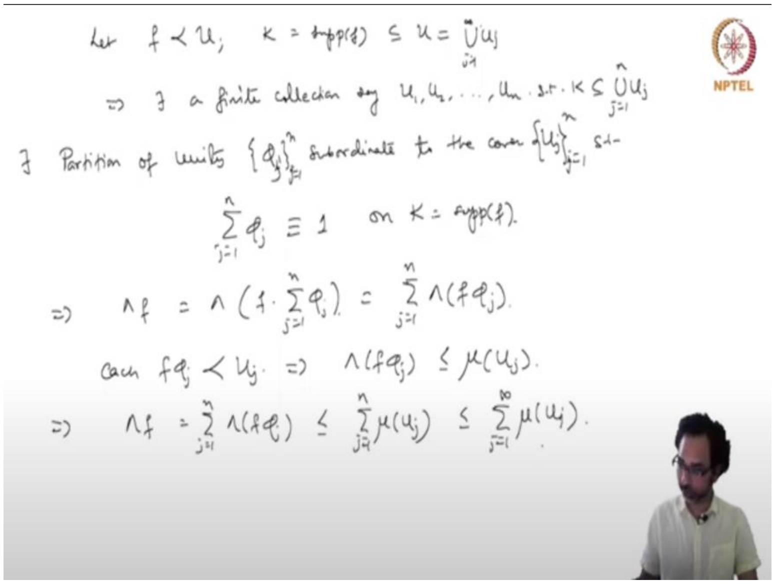

So let us fix a function which has compact support in u and whose range is between 0 and 1. And let k be the support of f. And this sets inside u now u is this union of u j's and so this is an open cover of compact set k and this implies that there exist a finite collection say u1 u2 up to u n such that k is contained in the union in the finite union of this n open sets u1 to un. And now I am going to use the existence of partition of unity phi i.

So there exist a partition of unity of phi i subordinate to the cover u j, j = 1 to n so j = 1 to, n here also. Such that summation phi j is identically equal to 1 on k so this is j = 1 to, n. So now this implies that lambda f can be written as lambda f times j = 1 to, n phi j because k was nothing but the support of f and so on the support of f this sum is 1. So f = f times sum of phi j j = 1 to, n. And by linearity this is j = 1 to, n lambda f times phi j.

Now each f phi j is supported inside u j is less than u j so this implies that lambda f phi j is less than or equal to mu of u j. And so this implies that lambda f is which is equal to j = 1 to, n lambda of, f phi j is less than or equal to the sum j = 1 to, n mu u j. And this is less than or equal to j = 1 to, n 1 to infinity mu u j. So this proves that this mu star is an outer measure.

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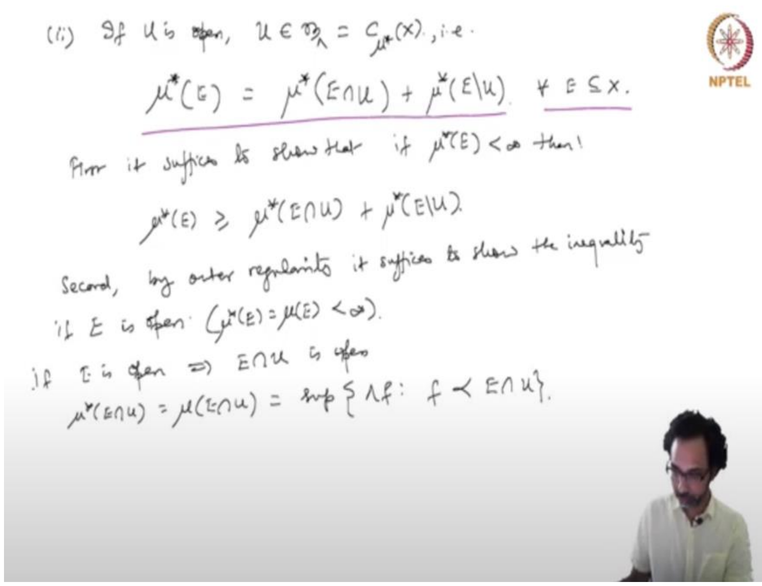

Now for the second claim we have to show that if u is open u belongs to the sigma algebra of caratheodary measurable subsets which is equivalently saying that mu star E = mu star E intersection u + mu star E - u for any E subset E in x. So it suffices first so we reduce the problem twice. So first it suffices to show that mu star E is greater than or equal to mu star E intersection u + mu star E - u because the other inequality is obvious due to mu star being an outer measure.

So by countable subadditivity mu star E is less than or equal to the right hand side. So we have to show that mu star is greater than or equal to the right hand side. On the other hand if mu star is infinite there is nothing to show. So we can only we can restrict our case our attention to the case if mu star E is finite then this holds. We can also make a second reduction which is that by outer regularity outer regularity it suffices to show this to show the inequality if E is open and still we have mu star E equals mu E if E is open this is finite.

So now we have to show when E is finite where E is open of finite measure we have to show this inequality. Now if E is open this implies that E intersection u is open and this means by our very definition of mu star. So mu star E intersection u is now mu of E intersection u because it is open. And now this is equal to supremum of lambda f such that f is less than E intersection u.

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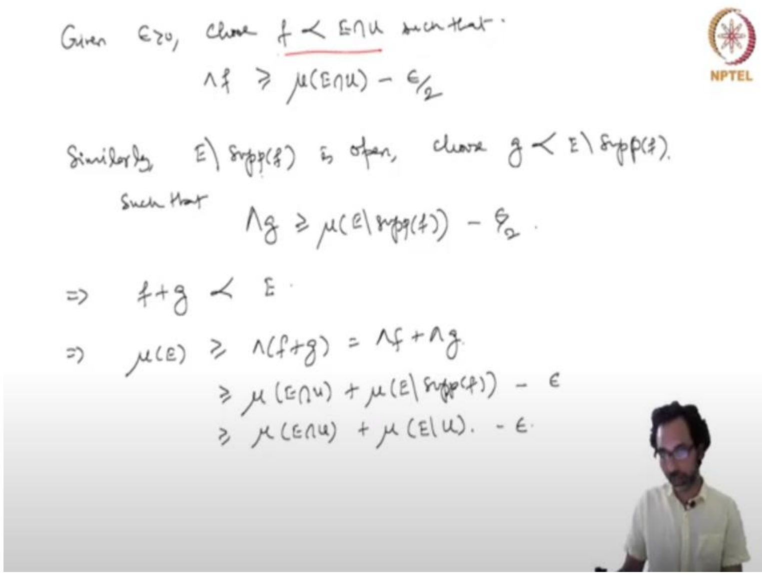

So now given epsilon greater than 0 choose f less than E intersection u such that lambda f + epsilon or rather lambda f is greater than or equal to mu E intersection u minus epsilon by 2. Similarly now that we have chosen a function of, f a compact support with support inside intersection u we have E minus support f is open. So choose g in less than E minus support f. Note that support f is compact. So it is closed.

So the compliment is open so this is why E minus support of, f is open. So now we can again choose g less than E minus support f such that lambda g is greater than or equal to mu of E minus support f minus epsilon by 2. But the way we have chosen f and g this implies that f + g is less than E because when g is 0 when f is non-zero g is 0 and vice versa. So f + g is still between 0 and 1 and its support is now between is now contained in E.

So this implies that mu E is greater than or equal to lambda times lambda of f + g which is equal to lambda f + lambda g by linearity and now we have chosen f and g so that this is greater than or equal to mu E intersection u + mu E minus support f minus epsilon by 2 and epsilon by 2 make an epsilon. But since support of, f is contained in source in u so this implies that this is greater than or equal to mu E intersection u + mu E - u minus epsilon.

So this is because support of a contained inside E intersection mu which is inside u. So mu of E - u is less than or equal to mu of E minus support of, f. So this shows that all open sets are indeed in the sigma algebra of caratheodary measurable sets B lambda. So this proves our secondary claim.

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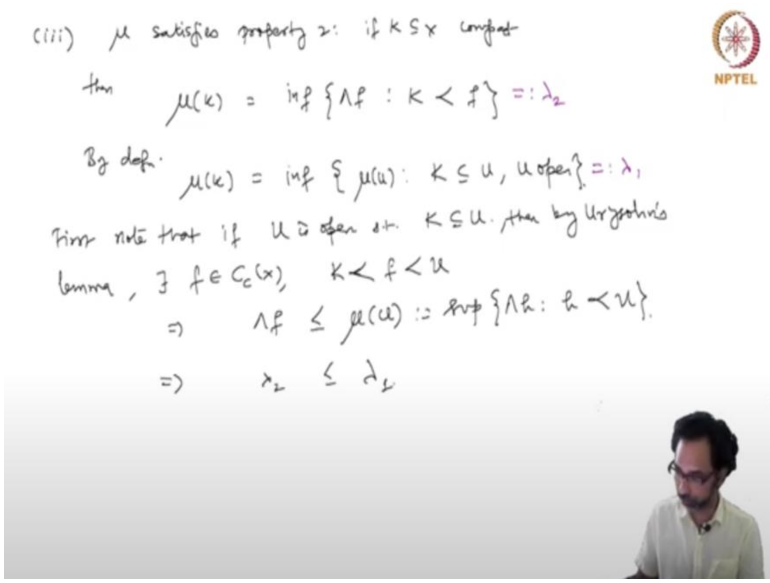

Now the third claim is that mu satisfies property to namely that if k is compact then mu k note that k is caratheodary measurable because it is a Borel set and Borel set lie in the sigma algebra of caratheodary measure x. So now we have to show that this is the infimum of lambda f such that k is less than f. Now recall that by definition we have that mu k is equal to the infimum of mu u such that k is contained in u and u is open.

So let me denote the right hand side here by lambda 1 and the right hand side here by lambda 2. And I am going to prove that lambda 1 = lambda 2. So, first note that if u is open such that k is contained in u then by Urysoh's lemma there exists a function f which is continuous with compact support such that k is less than f is less than u. And so this implies that lambda f is less than or equal to mu u because this latter on the right hand side was the supremum by definition this is the supremum of all such.

So let me write here such that h is less than u. So this means that lambda 2 is less than or equal to lambda 1. So now we will prove the reverse inequality.

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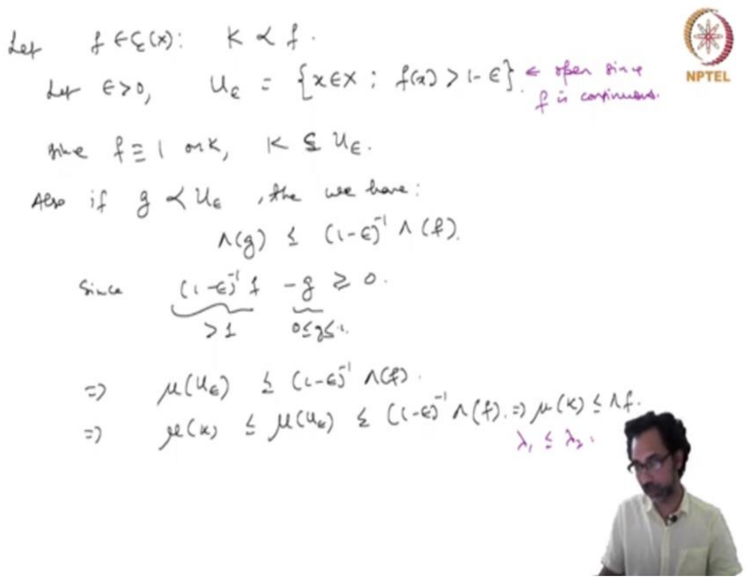

So let f in cc x such that k is less than f now we have to find an open set such that f is less than u epsilon. So how do we do this? So I said let epsilon greater than 0 and take u epsilon to be the set of all points in x such that f x is greater than 1 minus epsilon. So because f is continuous this set is open. Now we immediately have that since f = 1 on k, k is the subset of u epsilon and also if g is less than u epsilon.

Then we have that lambda g is less than or equal to 1 minus epsilon inverse times lambda f since because f x is greater than 1 minus epsilon. So this means that 1 minus epsilon inverse f - g is greater than or equal to 0 because this is strictly greater than 1 and this is between 0 and 1. So we have this inequality which shows that lambda g is less than or equal to 1 minus epsilon inverse times lambda f.

And now if we take the supremum on the left hand side this means that mu of u epsilon is less than or equal to 1 minus epsilon inverse lambda f and so letting epsilon goes to 0 this means on the right hand side on the right. Well first of all these means that mu k is less than or equal to mu u epsilon which is less than or equal to 1 minus epsilon inverse lambda f. And now we can take epsilon going to 0 on the right hand side so this implies that lambda k a mu k is less than or equal to lambda f.

So this shows that lambda 1 is less than or equal to lambda 2 and so these 2 quantities are in fact equal. So these this equality holds. So we have also proves the third assertion now we will derive some consequence of this third assertion.