Measure Theory

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Lecture – 4

Basic Properties of the Elementary Measure-Part 1

Key words: Elementary measure, properties of elementary measure.

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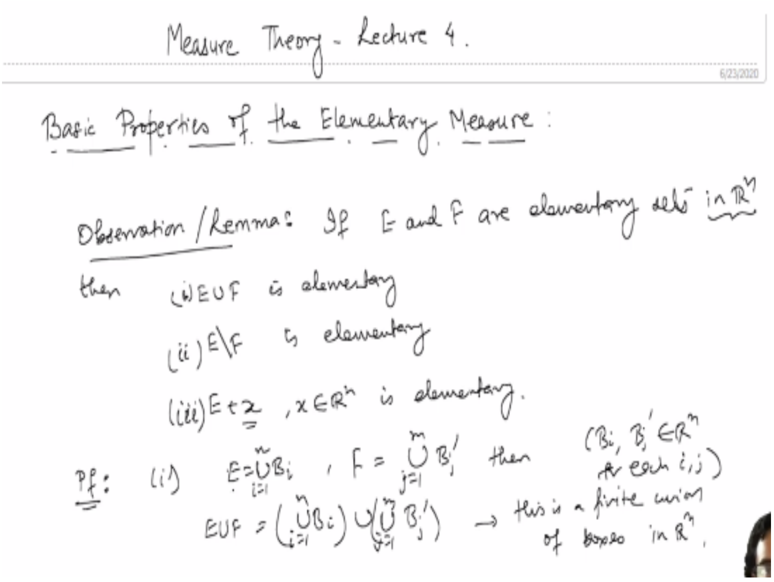

So, let us continue with our study of the basic properties of the elementary measure. Before we do that, I would like to state an easy observation or you can state it as a Lemma.

Observation / Lemma: Let us suppose that we are in $\mathbb{R}^n$. If E and F are elementary sets in $\mathbb{R}^n$, then

(i) $E \cup F$ is an elementary set.

(ii) $E \setminus F$ is an elementary set.

(iii) If E is translated by an element x in $\mathbb{R}^n$, then $(E + x)$ is also elementary.

This we have already seen before, but let us review it here. The proof is quite easy.

Proof: (i) So for the first part, it is very easy. Write E and F as a union of disjoint or non-disjoint boxes, that is,

$$E = \bigcup_{i=1}^{n} B_i ,$$

and

$$F = \bigcup_{j=1}^{m} B_j'$$

where $B_i, B_j' \in \mathbb{R}^n$ for each i, j. Then E union F is simply the union of the two things,

$$E \bigcup F = \left(\bigcup_{i=1}^{n} B_i \right) \bigcup \left(\bigcup_{j=1}^{m} B_j' \right)$$

and it is obvious that this is a finite union of boxes in $\mathbb{R}^n$. So it is quite obvious that the union of two elementary sets is elementary.

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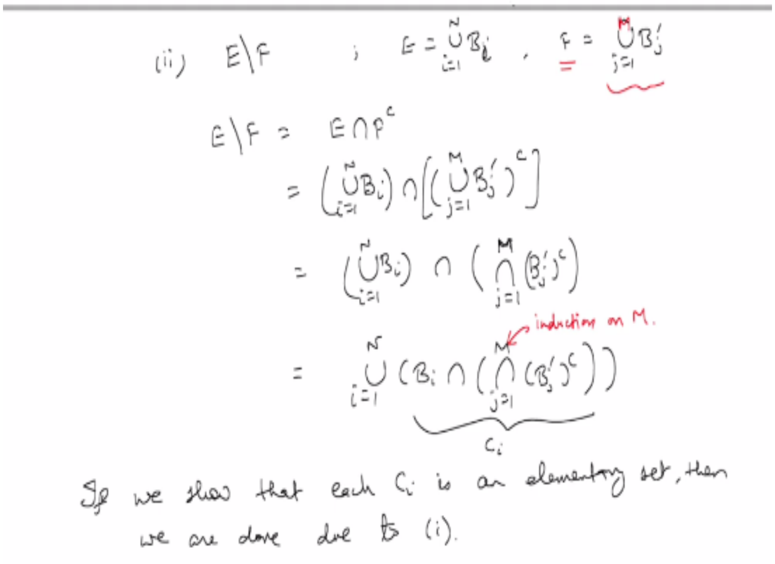

(ii) Now, let us come to the second one, that is $E \backslash F$. Let us again assume that

$$E = \bigcup_{i=1}^{N} B_i$$

and,

$$F = \bigcup_{j=1}^{M} B_j'.$$

such that B_i, B_j' are boxes in $\mathbb{R}^n$ for each i, j. Then $E \backslash F$ is simply E intersection by complement of F. So you can write it as follows.

$$E \backslash F = E \cap F^c$$

$$= \left(\bigcup_{i=1}^{N} B_i \right) \bigcap \left(\bigcup_{j=1}^{M} B_j' \right)^c$$

Now you can use De Morgan's law.

So you will get,

$$E \setminus F = \left(\bigcup_{i=1}^{N} B_i\right) \bigcap \left(\bigcap_{j=1}^{M} \left(B'_j\right)^c\right)$$

$$= \bigcup_{i=1}^{N} \left(B_i \bigcap \left(\bigcap_{j=1}^{M} \left(B'_j\right)^c\right)\right)$$

Let me call $\left(B_i \bigcap \left(\bigcap_{j=1}^{M} \left(B'_j\right)^c\right)\right)$ as C_i for $i = 1,...,N$. So if we show that each C_i is an elementary set, then we are done due to the first property of union of elementary sets being elementary.

So now let us try to show that each of these C_i's are elementary and we will do an induction on the number of boxes that are used to define the elementary set F. So if we can do an induction on M and show that for each M this is an elementary set, then we are done because then we will have a finite union of elementary sets, which is elementary. So let us try to do that.

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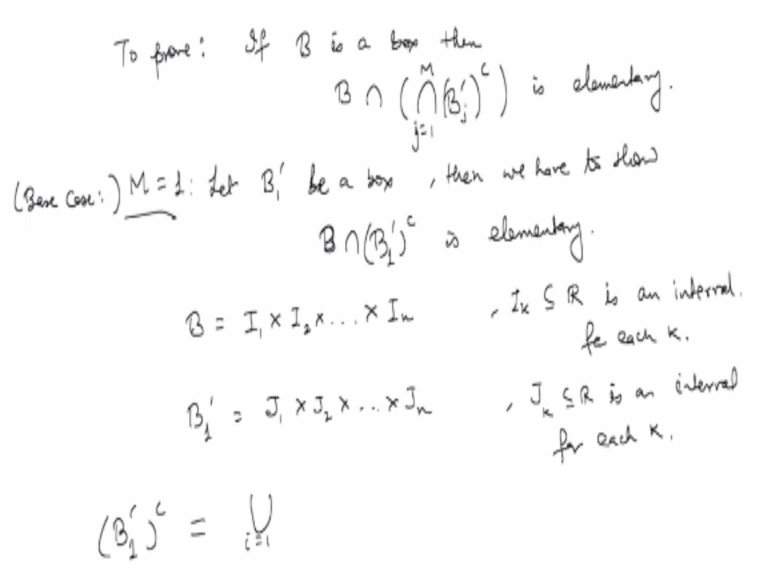

To prove: If B is a box then $\left(B \bigcap \left(\bigcap_{j=1}^{M} \left(B'_j\right)^c\right)\right)$ is elementary.

So first take $M = 1$, and this is our base case for the induction. Suppose B'_1 is a box. We have to show that $B \bigcap \left(B_1'\right)^{\mathrm{c}}$ is elementary.

So what we are going to do is to break it up into coordinates and we will see what happens for each coordinate. Remember that B is a box in $\mathbb{R}^n$, so it is a Cartesian product of intervals, that is

$$B = I_1 \times I_2 \times \ldots \times I_n\,,$$

where each I_k is a bounded interval in $\mathbb{R}$ for $k = 1,2,\ldots,n$. Similarly B_1' is a box. So this is a Cartesian product of let us say,

$$B_1' = J_1 \times J_2 \times \ldots \times J_n,$$

such that each J_k is a bounded interval in $\mathbb{R}$ for $k = 1,2,\ldots,n$.

Now we have to use some formula which describes the complement of a Cartesian product of n sets in terms of the complement of the individual coordinates sets. So I want to write down a formula for $\left(B_1'\right)^{\mathrm{c}}$ and this is as follows.

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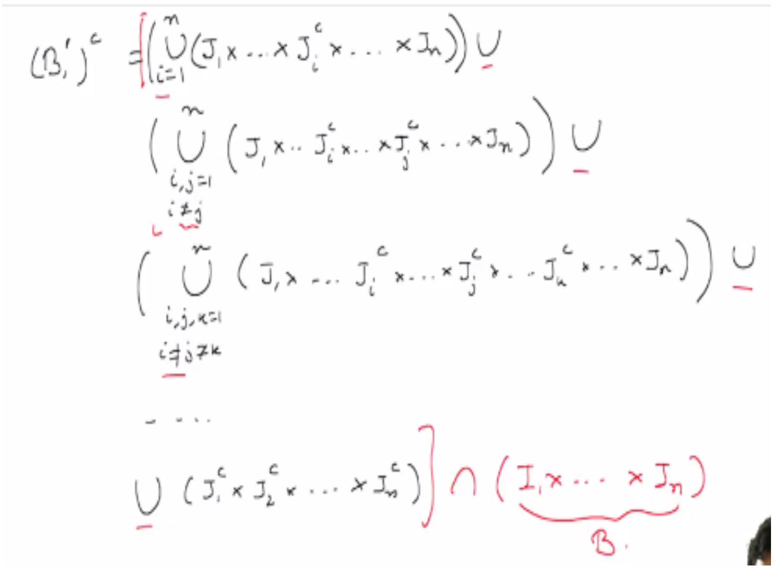

So we write $\left(B_1'\right)^{\mathrm{c}}$ as a union of components where first component is a union of $\left(J_1 \times J_2 \times \ldots \times J_i^{\mathrm{c}} \ldots \times J_n\right)$, where i varies from 1 to n. So the first component is a union of n sets with the j-th set with a compliment. Then we have the second component which is a union of $\left(J_1 \times J_2 \times \ldots \times J_i^{\mathrm{c}} \times \ldots \times J_j^{\mathrm{c}} \ldots \times J_n\right)$ over i and j and $i \neq j$. So now you will have two coordinates with complements and the rest remain the same. Here we are assuming $i \neq j$ because if $i = j$, then you will just have one set. So this is for distinct coordinates i and j and you will put a complement for each of the coordinates J_i and J_j. Similarly we get the third components for $i, j, k = 1,\ldots,n$ such that $i \neq j \neq k$ and three coordinates with the complement and the rest are the same and in the end you have the union of the sets where all the n coordinates are with a complement. That is,

$$\left(B_1'\right)^c$$

$$= \left(\bigcup_{i=1}^{n} \left(J_1 \times J_2 \times \ldots \times J_i^c \ldots \times J_n\right)\right) \bigcup \left(\bigcup_{\substack{i,\, j \\ i \neq j}} \left(J_1 \times J_2 \times \ldots \times J_i^c \times \ldots \times J_j^c \ldots \times J_n\right)\right)$$

$$\bigcup \left(\bigcup_{\substack{i,\, j,\, k \\ i \neq j \neq k}} \left(J_1 \times J_2 \times \ldots \times J_i^c \times \ldots \times J_j^c \ldots \times J_k^c \ldots \times J_n\right)\right) \ldots \bigcup \left(J_1^c \times \ldots \times J_n^c\right).$$

This formula is a bit tedious and might look scary, but the idea is very simple, and it just uses the elementary operations of unions and complements and cross products of Cartesian products. So I suggest you to go back and check what are the formulas when you have the Cartesian product of a union or the union of Cartesian products and so on and then what happens when you take the complements for such unions.

This is a union. So when we take the intersection of all these unions with our box B, then you can distribute this Cartesian product inside the unions and then you will have an intersection of two Cartesian products and then again you can distribute it. So let's write this down carefully.

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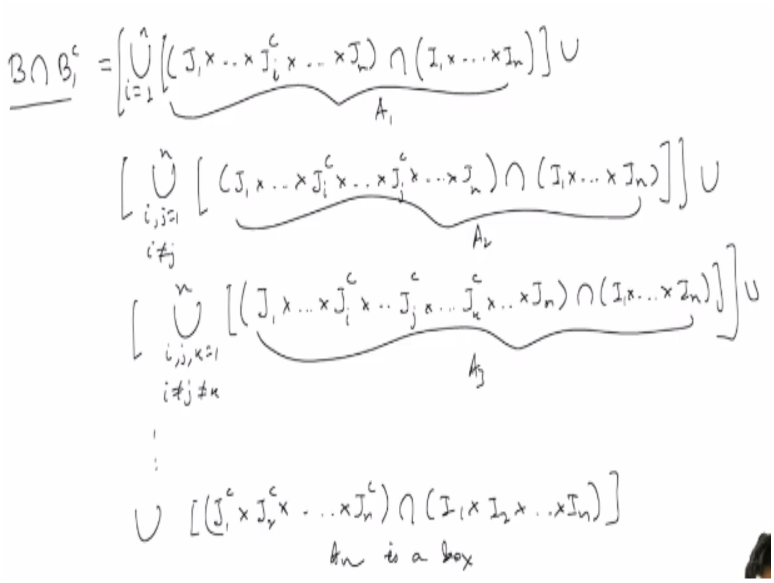

$$
\begin{aligned}
&B \bigcap \left(B_1'\right)^{\mathsf{c}} \\
&= \left[\bigcup_{i=1}^{n}\left[\left(J_1 \times J_2 \times \ldots \times J_i^{\mathsf{c}} \ldots \times J_n\right) \bigcap \left(I_1 \times I_2 \times \ldots \times I_n\right)\right]\right] \\
&\bigcup \left[\bigcup_{\substack{i,\, j \\ i \neq j}}\left[\left(J_1 \times J_2 \times \ldots \times J_i^{\mathsf{c}} \times \ldots \times J_j^{\mathsf{c}} \ldots \times J_n\right) \bigcap \left(I_1 \times I_2 \times \ldots \times I_n\right)\right]\right] \\
&\bigcup \left[\bigcup_{\substack{i,\, j,\, k \\ i \neq j \neq k}}\left[\left(J_1 \times J_2 \times \ldots \times J_i^{\mathsf{c}} \times \ldots \times J_j^{\mathsf{c}} \ldots \times J_k^{\mathsf{c}} \ldots \times J_n\right) \bigcap \left(I_1 \times I_2 \times \ldots \times I_n\right)\right]\right] \\
&\ldots \bigcup \left[\left(J_1^{\mathsf{c}} \times \ldots \times J_n^{\mathsf{c}}\right) \bigcap \left(I_1 \times I_2 \times \ldots \times I_n\right)\right].
\end{aligned}
$$

So you have for each of these an intersection of two sets each of which are Cartesian products themselves. So let us consider for example unions from the first group, let us call this set A_1.

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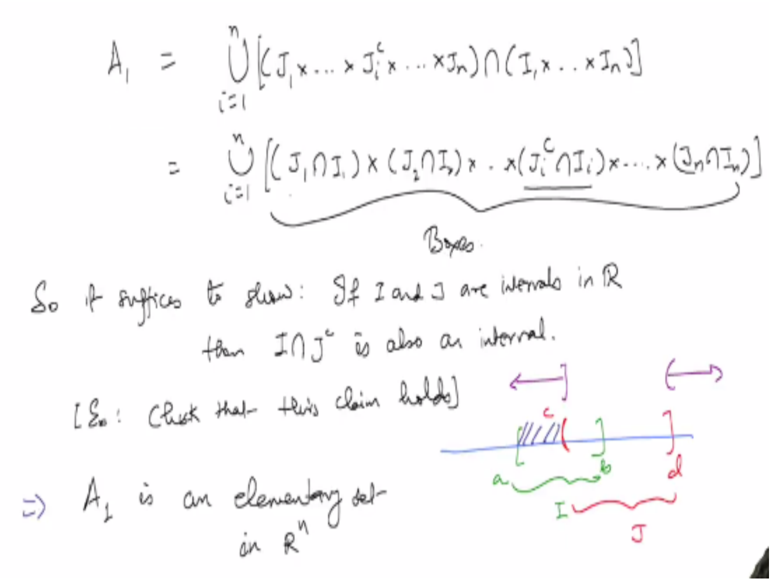

That is $A_1 = \bigcup_{i=1}^{n} \left[\left(J_1 \times J_2 \times \ldots \times J_i^c \ldots \times J_n \right) \bigcap \left(I_1 \times I_2 \times \ldots \times I_n \right) \right]$.

So this is just by the property of cross products and intersections,

$= \bigcup_{i=1}^{n} \left[\left(J_1 \bigcap I_1 \right) \times \left(J_2 \bigcap I_2 \right) \times \ldots \times \left(J_i^c \bigcap I_i \right) \ldots \times \left(J_n \bigcap I_n \right) \right]$.

So, now we have a finite union of such elements. So if we can prove that all these sets are boxes, then we will be done because then A_1will be an elementary set.

Similarly one could do for other groups where you have two complements, three complements or n complements and they will all be of the same form. So the idea is to show that if you have two intervals, then the intersection can either be empty or it will be another interval. If you can show that intersection of the complement of an interval with another interval is also an interval, then we would be done because then it will be a Cartesian product of intervals and so it will be a box. So, it suffices to show that if I and J are intervals in $\mathbb{R}$ then $I \cap J^c$ complement is also an interval or a union of intervals. So this is again quite easy and I leave it to you as an exercise, check that this claim holds.

So for example, I will just give a visual idea here.

So if you have the interval I from a to b and you have interval J from c to d as above. So J^c is the region left of c and right of d. Then $I \cap J^c$ is simply the region from a to c. So similarly you can check all other cases. If I and J are disjoint intervals, then $I \cap J^c$ will simply be I. In all other cases $I \cap J^c$ will be an interval. It is very easy to prove, and I leave it as an exercise for you.

Therefore, we have that all the constituent sets in the union of A_1 is given by boxes. Therefore, A_1 is an elementary set because it is a finite union of boxes. Now we can go back to the other groups $A_2, A_3, \ldots, A_n$. Since you have the intersection coordinate wise, therefore each one will be an interval and each of these constituent sets will be a box and therefore you will have a finite union of boxes. In particular A_n itself is a box because you have just one constituent set. Therefore we have proved that $B \bigcap \left(B_1'\right)^c$ is an elementary set. So this proves the base case for our induction.

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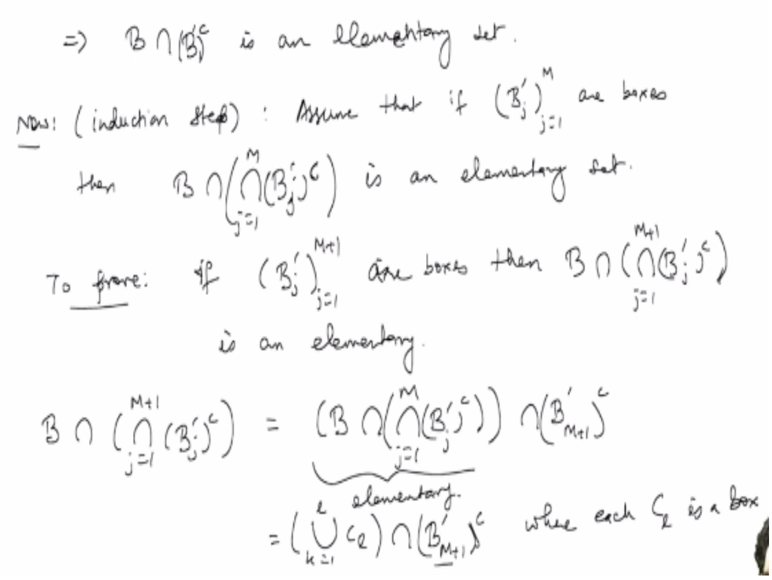

Now we come at the induction step. Assume that if B_j' 's are boxes for $j = 1,2,\ldots,M$, then $\left(B \bigcap \left(\bigcap_{j=1}^{M} \left(B_j'\right)^c\right)\right)$ is an elementary set. We will have to prove that if B_j' 's are boxes for $j = 1,2,\ldots,M+1$, then $\left(B \bigcap \left(\bigcap_{j=1}^{M+1} \left(B_j'\right)^c\right)\right)$ is an elementary set. This will prove our induction hypothesis. So, let us write down what is $\left(B \bigcap \left(\bigcap_{j=1}^{M+1} \left(B_j'\right)^{c}\right)\right)$. Note that you can take out M of these $\left(B_j'\right)^{c}$ and group it with B. So,

$$B \bigcap \left(\bigcap_{j=1}^{M+1} \left(B_j'\right)^{c}\right) = \left(B \bigcap \left(\bigcap_{j=1}^{M} \left(B_j'\right)^{c}\right)\right) \bigcap \left(B_{M+1}'\right)^{c}.$$

Now from our induction hypothesis we know that, $\left(B \bigcap \left(\bigcap_{j=1}^{M} \left(B_j'\right)^{c}\right)\right)$ is an elementary set. This is in fact a finite union of boxes, let us call it $\bigcup_{k=1}^{l} C_k$, where each C_k is a box. Now we have to take $\left(\bigcup_{k=1}^{l} C_k\right) \bigcap \left(B_{M+1}'\right)^{c}$.

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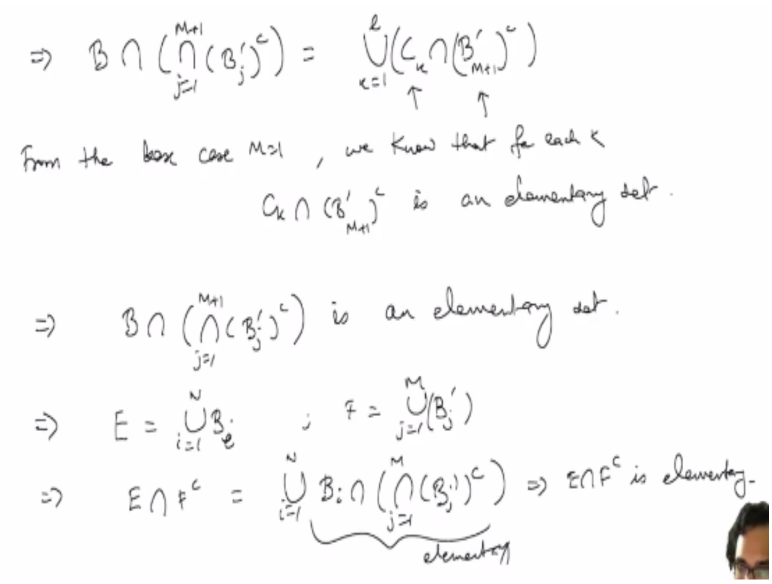

So now we can distribute it inside the union. This implies

$$B \bigcap \left(\bigcap_{j=1}^{M+1} \left(B_j'\right)^{c}\right) = \bigcup_{k=1}^{l} \left(C_k \bigcap \left(B_{M+1}'\right)^{c}\right).$$

So we are just distributing the intersection inside the union and now this is a box. We have already proved from the base case $M = 1$ that for each k, $\left(C_k \bigcap \left(B_{M+1}'\right)^{c}\right)$ is an elementary set. Now we have again a finite union of elementary sets. This implies $B \bigcap \left(\bigcap_{j=1}^{M+1} \left(B_j' \right)^{\mathsf{c}} \right)$ is an elementary set. This implies if $E = \bigcup_{i=1}^{N} B_i$, and $F = \bigcup_{j=1}^{M} B_j'$, then

$$E \bigcap F^{\mathsf{c}} = \bigcup_{i=1}^{N} B_i \bigcap \left(\bigcap_{j=1}^{M} \left(B_j' \right)^{\mathsf{c}} \right).$$

Each of $\left(B_i \bigcap \left(\bigcap_{j=1}^{M} \left(B_j' \right)^{\mathsf{c}} \right) \right)$ is elementary, so is the union, and hence $E \bigcap F^{\mathsf{c}}$ is elementary.

This completes our proof.