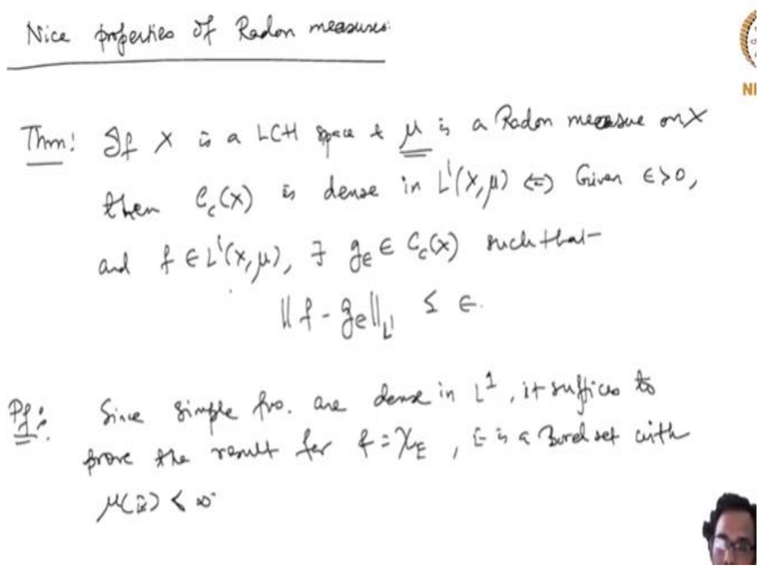
Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Module No # 12
Lecture No # 59
Properties of Radon measures and Lusin's theorem on LCH spaces

(Refer Slide Time: 00:22)

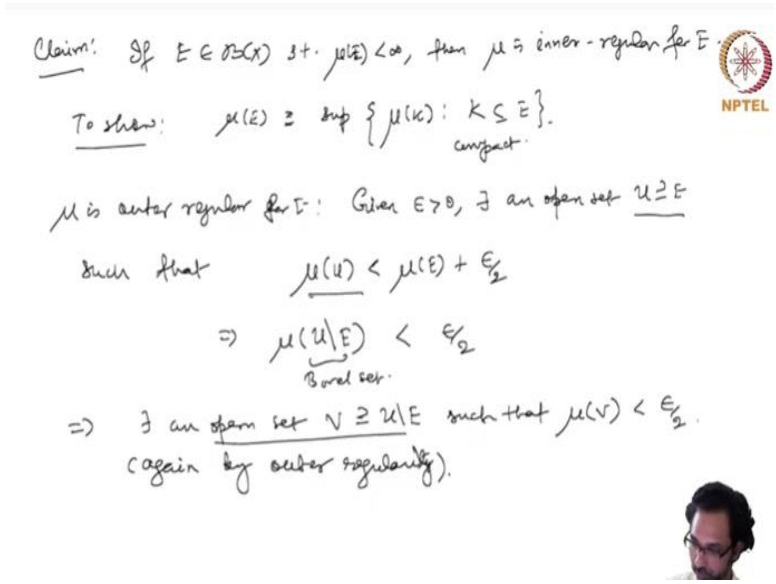


So now let us look at some nice properties that Radon measures enjoy nice properties of Radon measures. So first is that so let me write this as theorem if X is a locally compact Hausdorff theorem space and mu is a Radon measure on X. Then the space of continuously compactly supported functions is dense in L1 X mu meaning that given any epsilon greater than 0. And f in L1 X mu there exist g epsilon so this g depends on epsilon which is a continuous function with compact support.

Such that the norm of f – g epsilon of L1 norm this is less than or equal to epsilon. So we have already seen that in Rd with Lebesgue measure this kind of results holds that CCRd is dense in L1 of Rd with the Lebesgue measure. But now if you have a Radon measure mu then also this kind of results holds. So let us see a proof the first thing to note is that since simple functions are dense in L1 just by the definition of the Lebesgue integrals it is suffices to prove the result for f equals the indicative function E where E is a Borel set with finite measure.

So, mu E is finite so if we can prove this for indicative functions of Borel sets with finite measure then we can easily do it for any general L1 measure by first taking finite linear combination of simple function and then approximating in L1 norm by simple functions.

(Refer Slide Time: 03:28)



So now we need another result I claim that if E is a Borel set such that mu E is finite then mu is inner regular for E. So note that Radon measure assumption is only that mu is in a regular for open sets but I now claim that any Borel set with finite measure mu is inner regular. So let us see why this is true? So we have to show that mu of E is equal to the supremum of mu k such that k is compact in E.

So to do this first we use that E is a Borel set so mu is outer regular for E which means that given epsilon greater than 0. There exist an open set u containing in E such that we have mu u is less than mu E + epsilon by 2. So this is just by outer regularity of mu for Borel sets so this means that mu of u –E is less than epsilon by 2 and in particular this is a Borel set. So this implies that there exist an open set V containing u – E such that mu V is less than epsilon by 2.

Again by outer regularity so we have first chosen an open set containing E such that mu E is less than mu E + epsilon by 2. And then we choose an open set v containing u – E such that mu v is less than epsilon by 2.

(Refer Slide Time: 06:42)

By inner regularity for open sets : μ is inner-regular for U:

$\exists\ K \subseteq U$ / K compact such that

$$\mu(U) < \mu(K) + \epsilon/2 \quad \text{— ①.}$$

Now, take $F = K \setminus \underbrace{V}_{\text{open set}} = K \cap V^c \leftarrow$ compact

$$F \subseteq E. \quad \text{since} \quad V \supseteq U \setminus E.$$

and we have

$$\mu(F) = \mu(K) - \mu(V) \overset{①}{>} \mu(U) - \epsilon/2 - \mu(V).$$

$$\overset{\text{Monotonicity}}{>} \mu(E) - \epsilon/2 - \epsilon/2$$

$$= \mu(E) - \epsilon.$$

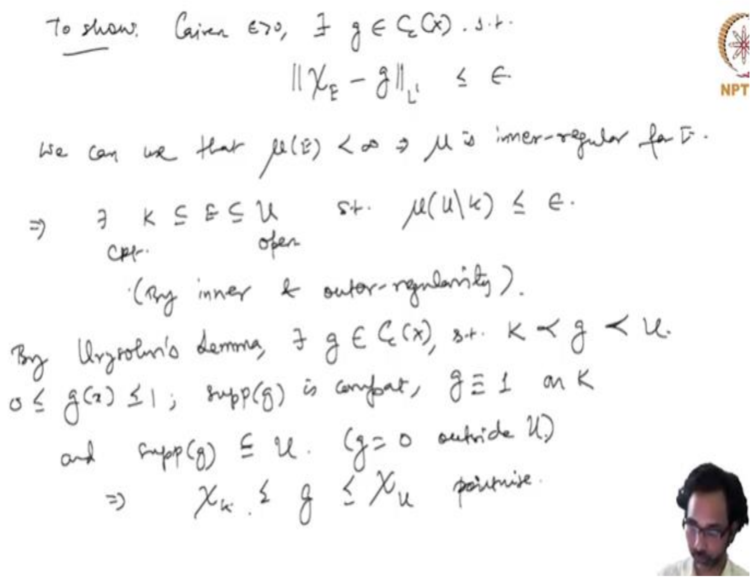
$$\Rightarrow \mu(E) = \sup\{\mu(F) : F \subseteq E,\ F \text{ compact}\}.$$

Now by inner regularity for open sets we have that mu is inner regular for u which means that there exist a compact set k compact such that mu of u is less than mu k + epsilon by 2. Now take F to be the set k – v so remember that v was an open set so this is equal to k intersection v complement and since k is compact and v complement is closed so this is compact. So this set is compact and F is a subset of E this is because v covers u – v is u – E is contained in v.

So F is in fact contained inside E and finally we have that the measure of mu of F is equal to mu k – mu v. But mu k we can use this inequality so from 1 this is 1 which says that mu k is greater than mu u – epsilon by 2 –mu v. But mu v is but mu u is greater than mu E by monotonicity because E is the subset of u and then we have because mu v is less than epsilon by 2 this is greater than mu E – epsilon by 2 – epsilon by 2 which is mu E – epsilon.

So this means that we have found a compact set which is greater than mu v – epsilon which means that mu E is the supremum of mu f such that F in E is compact. So once we have shown this it is easy to show that we can find a continuous function with compact support which approximates he indicative function of Borel set in the L1 norm.

(Refer Slide Time: 10:21)



So now we have to show that there exist given epsilon there exist g in CCX such that the L1 norm of Chi E – g is less than or equal to epsilon. So now we can use the fact that mu E is finite implies that mu is inner regular for E and this means that there exist a compact set k in E and an open set u. So this is compact and this is open such that mu u – k is less than or equal to epsilon because by inner and outer regularity.

So this is by inner and outer regularity and then we can find by Urysohn's lemma there exist, a continuous function with compact support. Such that k less than g less than u which means so in this in our last lecture we use this notation to mean that g takes values between 0 and 1. Support of g is compact g is identically equal to 1 on k and support of g equals is a subset of u. So g is 0 outside u so by Urysohn's lemma we have this and we have seen that this is equivalent to.

Saying that the indicative function of k is less than or equal to g and this is less than or equal to indicative function of u point wise. Meaning that g can be sandwiched between the indicative function of k and the indicative of u.

(Refer Slide Time: 13:11)

$$\|\chi_E - g\|_{L^1} = \int_X |\chi_E - g|\, d\mu.$$

$$\le \int_X |\chi_u - g|\, d\mu \quad (\text{since } \chi_E \le \chi_u).$$

$$= \int_X (\chi_u - g)\, d\mu.$$

$$\le \int_X (\chi_u - \chi_k)\, d\mu = \mu(u) - \mu(k) \le \epsilon.$$

So now we can estimate Chi E – g the L1 norm and this is nothing but the integral of Chi E – g mod d mu and this is less than or equal to integral of x over x Chi u – g d mu. Since Chi E is less than or equal to Chi u and this is nothing but now Chi u is always greater than or equal to g. So we can lose the modulus sign so this is equal to Chi u – g d mu. And now this is again less than or equal to Chi u – Chi k d mu.

And then this is nothing but the mu of u minus the mu of k and this is less than or equal to epsilon. So we see that continuous functions of contact support are dense in the L1 norm are dense in the L1 functions in the space of L1 functions on x with the Radon measure.

(Refer Slide Time: 14:52)

<u>Corollary</u>: (Lusin's theorem): If X is a LCH space and μ is a Radon measure on X, and if f is a Borel measurable fn. on X, $f: X \to \mathbb{C}$ such that $f = 0$ outside a set $E \in \mathcal{B}(X)$ of finite measure. Then, given $\epsilon > 0$, $\exists$ a Borel set $A \in \mathcal{B}(X)$ s.t. $\mu(A) \le \epsilon$ and f is continuous on $E \setminus A$.

<u>Pf</u>: Repetition of the proof for $\mathbb{R}^d$. [Ex: Check these details]. ∎

An so once we have this as a corollary we have Lusin's theorem which says that if X is locally compact Hausdorff space and mu is a Radon measure on X. And if f is a Borel measureable function on X. So f is a complex valued Borel measureable function then I need another assumption that such that f is 0 outside a set E a Borel set E of finite measure. So again we would need an inner regularity so outside a set E of finite measure f is 0.

So support of, f is contained inside a set of finite measure given any epsilon greater than 0 there exist a Borel set A such that the measure of A is less than or equal to epsilon and f is continuous on E – A. So this is a generalization of the Lusin's theorem that we have seen for Rd to locally compact Hausdorff spaces but with a radon measure and the proof is essentially just a repetition of the proof for Rd.

But taking into account that we have a radon measure and we have chosen f to be to have support inside a set of finite measure and then we can basically repeat the proof. But this is left as an exercise to check the details of this proof so then we get Lusin's theorem for locally compact Hausdorff spaces with radon measures.